PROBABILITY THEORY AND STOCHASTIC PROCESSES (20A54403)

LECTURE NOTES

II-B.TECH & I-SEM

Department of Electronics and Communication Engineering

SREE VENKATESWARA COLLEGE OF ENGINEERING
NAAC 'A' Accredited Institution
An ISO 9001:: 2015 Certified Institution
(Approved by AICTE, New Delhi and Affiliated to JNTUA, Ananthapuramu)
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Course Code	SIGNALS AND SYSTEMS	L	T	P	C
20A04301T		3	0	0	3
Pre-requisite	Mathematics - I	Semester			III

Course Objectives:

- To gain the knowledge of the basic probability concepts and acquire skills in handling situations involving more than one random variable and functions of random variables.
- To understand the principles of random signals and random processes.
- To be acquainted with systems involving random signals.
- To gain knowledge of standard distributions that can describe real life phenomena

Course Outcomes (CO): After completion of the course, the student can able to

CO_I: Understand the fundamental concepts of probability theory, random variables and evaluate probability Distribution and Density Functions for different random variables.

CO_II: Analyze the random variable by calculating statistical parameters

CO_III: Analyze multiple random variables by calculating different statistical parameters and understand the linear transformation of Gaussian random variable

CO_IV: Analyze the random process in both time and frequency domain.

CO_V: Derive the response of linear system for random signals as input and explain the low pass and band pass noise models of random process

Unit – I: Probability & Random Variable

Probability through Sets and Relative Frequency: Experiments and Sample Spaces, Discrete and Continuous Sample Spaces, Events, Probability Definitions and Axioms, Mathematical Model of Experiments, Probability as a Relative Frequency, Joint Probability, Conditional Probability, Total Probability, Bayes' Theorem, Independent Events, Problem Solving.

Random Variable: Definition of a Random Variable, Conditions for a Function to be a Random Variable, Discrete, Continuous, Mixed Random Variable, Distribution and Density functions, Properties, Binomial, Poisson, Uniform, Gaussian, Exponential, Rayleigh, Conditional Distribution, Methods of defining Conditioning Event, Conditional Density, Properties, Problem Solving.

Unit – II: Operations on Random variable

Operations on Single Random Variable: Introduction, Expectation of a random variable, moments-moments about the origin, Central moments, Variance and Skew, Chebyshev's inequality, moment generating function, characteristic function, transformations of random variable.

Multiple Random Variables: Vector Random Variables, Joint Distribution Function, Properties of Joint Distribution, Marginal Distribution Functions, Conditional Distribution and Density – Point Conditioning, Interval conditioning, Statistical Independence, Sum of Two Random Variables, Sum of Several Random Variables, Central Limit Theorem, (Proof not expected), Unequal Distribution, Equal Distributions.

Unit – III: Operations on Multiple Random variables

.**Operations on Multiple Random Variables:** Expected Value of a Function of Random Variables, Joint Moments about the Origin, Joint Central Moments, Joint Characteristic Functions, Jointly Gaussian Random Variables: Two Random Variables case, N Random Variable case, Properties of Gaussian random variables, Transformations of Multiple Random Variables, Linear Transformations of Gaussian Random Variables.

Unit – IV: Random Processes

Random Processes-Temporal Characteristics: The Random Process Concept, Classification of Processes, Deterministic and Nondeterministic Processes, Distribution and Density Functions, concept of Stationarity and Statistical Independence, First-Order Stationary Processes, Second-Order and Wide-Sense Stationarity, N-Order and Strict-Sense Stationarity. Time Averages and Ergodicity, Mean-Ergodic Processes, Correlation-Ergodic Processes, Autocorrelation Function and Its Properties, Cross-Correlation Function and its Properties, Covariance Functions, Gaussian Random Processes, Poisson Random Process.

Random Processes-Spectral Characteristics: The Power Density Spectrum and its Properties, Relationship between Power Spectrum and Autocorrelation Function, The Cross-Power Density Spectrum and its Properties, Relationship between Cross-Power Spectrum and Cross-Correlation Function.

Unit – V:	Random Signal Response of Linear Systems

Random Signal Response of Linear Systems: System Response – Convolution, Mean and Mean squared Value of System Response, autocorrelation Function of Response, Cross-Correlation Functions of Input and Output, Spectral Characteristics of System Response: Power Density Spectrum of Response, Cross-Power Density Spectrums of Input and Output, Band pass, Band Limited and Narrowband Processes, Properties.

Noise Definitions: White Noise, colored noise and their statistical characteristics, Ideal low pass filtered white noise, RC filtered white noise.

Textbooks:

1. Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
2. Athanasios Papoulis and S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002

Reference Books:

1. Simon Haykin, "Communication Systems", 3rd Edition, Wiley, 2010.
2. Henry Stark and John W.Woods, "Probability and Random Processes with Application to Signal Processing," 3rd Edition, Pearson Education, 2002.
3. George R. Cooper, Clave D. MC Gillem, "Probability Methods of Signal and System Analysis," 3rd Edition, Oxford, 1999.

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CHAPTER 1

Introduction to Probability

1.1 Introduction

In communication, signals are broadly classified into two types: Deterministic signals and Non-deterministic signals.

1.1.1 Deterministic signal

The value of the signal can be determined at any instant of time. The deterministic signals convey no information.

Example: $M(t) = 10 \times t$

1.1.2 Non-Deterministic signal

The value of the signal cannot be determined at any instant of time. This signal carry the information. It is also called as random signals or statical signals.

Examples:

- Unpredictable information signals
  - — Audio signal, Video signal, Voice signal, Image signal
- Noise generated in the receiver and channel
- 0's and 1's generated by the computer
- stock market

Communication is a process of conveying information from one point to another point . Communication system consists of three major blocks as shown in Fig. 1.1

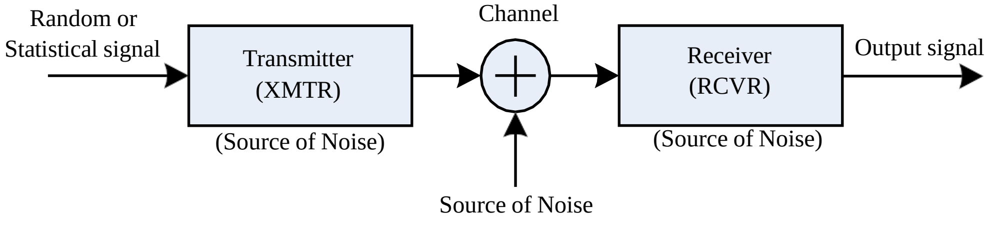


Fig. 1.1 Basic communication system

The transmitter (XMTR) and receivers (RCVR) are sources of noise, which is generated by a resistor, diode, transistor, FETs, etc., These components are used in XMTR's and RCVR's. Channel is major source of noise, which are man-made noise, interference from other XMTR's etc., To analyze of information signal and noise, the probability concepts are used. Analysis means calculation of power, energy and frequency etc.,

1.2 Basics in Probability (or) Terminology in Probability

1.2.1 Outcome

The outcome is an end result of an experiment.
Examples:

- Getting Head or Trail in tossing a coin.
- Getting 1, 2, 3, 4, 5, 6 in throwing a dice.

1.2.2 Trail

It is the single performance of random experiment.
Example:

- One attempt of rolling dice
- One attempt of tossing a coin

1.2.3 Random experiment

An experiment whose outcome s are not known in advance.
Example:

- Tossing a coin.
- Rolling a dice.
- Measuring a noise voltage at the terminals of the system.

1.2.4 Random event

A random event is an outcome or set of outcomes of an random experiment that share a common attribute.
Example:

- In a rolling a die getting even number or odd number are called as random event.
  i.e., Total outcome $S = \{1, 2, 3, 4, 5, 6\}$
  Getting even number $A_e = \{2, 4, 6\}$
  Getting odd number $A_o = \{1, 3, 5\}$

1.2.5 Certain event

If the probability of an event is equal to one (1), then it is caller certain event.
Example: Rising the sun in the East.

1.2.6 Impossible event

If the probability of an event is equal to zero (0). Ex: Rising of sun in the West.

1.2.7 Elementary event

The single outcome of an random experiment is called elementary event.
Example: Getting Head in tossing a coin

1.2.8 Null event

If there is common element between two events then it is called null event.

Example: In a rolling a dice the total outcome $S = \{1, 2, 3, 4, 5, 6\}$
Getting even number $A_e = \{2, 4, 6\}$
Getting odd number $A_o = \{1, 3, 5\}$
$\therefore A_e \cap A_o = \varphi$

1.2.9 Mutually exclusive event

The two events A and B are said to be mutually exclusive, If they have no common element. Example: In a rolling a dice the total outcome $S = \{1, 2, 3, 4, 5, 6\}$
Getting even number $A_e = \{2, 4, 6\}$
Getting odd number $A_o = \{1, 3, 5\}$
Getting numbers less than 4, i.e., $A_4 = \{1, 2, 3\}$ $\therefore A_e \cap A_o = \varphi$
A_e and A_o $-\!\!\rightarrow$ are mutually exclusive event.
$A_e \cap A_4$ $-\!\!\rightarrow$ are not mutually exclusive event.

1.2.10 Equally Likely event

If the probability of occurrence of events are equal then they are called likely events.

Example: In a rolling a dice the total outcome $S = \{1, 2, 3, 4, 5, 6\}$
Getting even number $A_e = \{2, 4, 6\}$; $P(A_e) = 3/6 = 1/2$
Getting odd number $A_o = \{1, 3, 5\}$; $P(A_e) = 3/6 = 1/2$
So, $P(A_e)$ and $P(A_o)$ are likely event.

Getting numbers less than 5, i.e., $A_5 = \{1, 2, 3, 4\}$ then $P(A_5) = 4/6 = 2/3$. So,this not likely event.

1.2.11 Exhaustive event

The total number of outcomes of an random experiment is called exhaustive event.
Example: In a rolling a dice consisting of '6' outcomes.
In a rolling two dices, the exhaustive events are '36'.

1.2.12 Union of a event

The union of two events A and B is the set of all outcomes, which belongs to A or B or both. Example: A_e or $A_5 = A_e + A_5 = A_e \cup A_5 = \{1, 2, 3, 4, 5, 6\}$

1.2.13 Intersection of an event

It is the common elements between A and B events. Example: $A_e \cap A_5 = \{2, 4\}$

1.2.14 Complement of an event

It is the complement of an event A is the event containing all the point in 'S', but not in 'A'. Example: In a rolling a dice the total outcome $S = \{1, 2, 3, 4, 5, 6\}$
Getting numbers less than 5, i.e., $A = \{1, 2, 3, 4\}$
Complement of A is $\bar{A} = \{5, 6\}$

1.2.15 Sample space

The set of possible outcomes of an random experiment is called sample space.

Example:

- In tossing a coin, sample space is $\{H, T\}$
- Sample space in rolling a dice, $S = \{1, 2, 3, 4, 5, 6\}$
- The sample space of a random experiment in which a dice and coin are tossed.
  $S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),$
  $(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$
- Sample space of tossing two coins: $S = \{HH, HT, TH, TT\}$
- Sample space of two dice rolled.
  $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$
  $(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)...$
  $(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Types of sample space

1. Discrete sample space: If the sample space consists of discrete set of samples then the sample space is said to be discrete sample space.It is two types.
   - Finite sample space: $S = \{T, H\}$ in tossing a coin $-\rightarrow$ finite.
   - Infinite sample space: Finding odd or even or integer number $-\rightarrow$ infinite.
2. Continuous sample space: If the sample space contain infinite number of outcomes or sample space then it is called continuous sample space.
   Example: Finding real number
   The probability of sample space is always equal to one.

1.2.16 Difference

The set consisting of all elements of 'A' which do not belongs to 'B' is called the difference of A and B. It is denoted by $A - B$.

1.3 Definition of Probability

Mostly used definitions are

1. Relative frequency approach $\longrightarrow$ Experiment
2. Classical approach $\longrightarrow$ Theory
3. Approximate approach $\longrightarrow$ Theory

1.3.1 Relative frequency approach

It is based on experimentation or practical. The probability of an event (or outcome) is the proportion of times the event would occur in a long run of repeated experiments.

Suppose if the random experiment is performed 'n' number of times then event A has occurred n_A times then probability of event A can be written as

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}; \quad 0 \le P(A) \le 1 \tag{1.1}$$

It is also known as a *posteriori probability*, i.e., the probability determines after the event.

Consider two events A and B of the random experiment. Suppose we conduct 'n' independent trails of this experiment and events A and B occurs in $n(A)$ and $n(B)$ trails vice versa. Hence, the event $A \cup B$ or $A + B$ or $P\,(A \text{ } or \text{ } B)$ occurs in $n(A) + n(B)$ trails and

$$\begin{aligned} P(A \cup B) &= P(A + B) \\ &= \lim_{n\to\infty} \frac{n_A + n_B}{n} \\ &= \lim_{n\to\infty} \frac{n_A}{n} + \lim_{n\to\infty} \frac{n_B}{n} \\ &= P(A) + P(B) \end{aligned}$$

where, $n \longrightarrow$ number of times experiment performed

$n_A \longrightarrow$ number of times event A occurred; $n_B \longrightarrow$ number of times event B occurred.

$$\therefore\ P(A \cup B) = P(A + B) = P(A) + P(B) \tag{1.2}$$

The equation (1.2) gives that, A and B are mutually exclusive event. If they are not mutually exclusive then, it is given in equation (1.3).

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A + B) &= P(A) + P(B) - P(AB) \end{aligned} \tag{1.3}$$

Example: An experiment is repeated number of times as shown in below. Find the probability of each event.

Random Experiment	Getting Head
1	1
10	6
100	50

Solution: Relative frequency:

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}; \quad 0 \leq P(A) \leq 1$$

$$P(A) = \frac{M}{N} = \frac{1}{1} = 1; \qquad P(B) = \frac{M}{N} = \frac{6}{10}; \qquad P(C) = \frac{M}{N} = \frac{50}{100}$$

1.3.2 Classical approach

It is based on theoretical approach, which means without experimentation. Here the number of total outcomes of an random experiment is calculated and probability of event A is calculated by finding number of favourable outcomes of event A.

i.e., $P(A) = \frac{n_A}{n}$

where $n \longrightarrow$ total number of sample points in sample space.

$n_A \longrightarrow$ number of favourable outcomes to event A.

Example: Tossing a coin. $S = \{T, H\}$. Here $n = 2$ and $P(H) = P(T) = \frac{1}{2}$

1.3.3 Approximate or Axiomatic approach

It is based on the axioms of theorems. Let us consider 'S' be the sample space consisting all possible outcomes an experiment. The events $A, B, C...$ are subsets of sample space. The function $P(.)$ defines which associates with event 'A' is a real number called probability of A. This function $P(.)$ has to satisfies the following axioms.

Axiom 1. Non-negativity: For every event 'A'; $0 \leq P(n) \leq 1$

Axiom 2. Certainty or normalization: For sure or certain events; $P(S) = 1$

Axiom 3. Additivity: If A and B are mutually exclusive events;
$P(A + B) = P(A) + P(B)$

1.3.4 Probability Measure, Theorems

Theorem 1.3.1. *If φ is an empty set then* $P(\varphi) = 0$

Proof. Let 'A' be any set such that A and φ are mutually exclusive. i.e., $A + \varphi = A$. Using Axiom 3,

$$P(A + \varphi) = P(A)$$

$$P(A) + P(\varphi) = P(A)$$

$$\therefore\ P(\varphi) = 0.$$

□

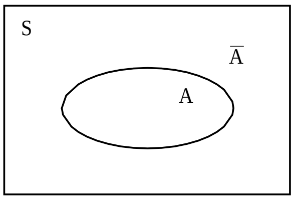
Theorem 1.3.2. *In sample space 'S' such that* $B = A + \overline{A}$; $P(\overline{A}) = 1 - P(A)$.

Proof. The sample space S can be divided into two mutually exclusive events A and $\overline{A}$ as shown in Venn diagram.

$$P(S) = 1$$

$$P(A) + P(\overline{A}) = 1$$

$$\therefore\ P(\overline{A}) = 1 - P(A).$$



□

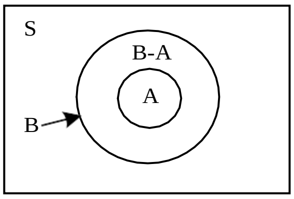
Theorem 1.3.3. *If* $A \subset B$ *then* $P(A) \leq P(B)$

Proof. If $A \subset B$, then B can be divided into two mutually exclusive events A and B as shown in Venn diagram. Thus,

$$P(B) = P(A) + P(B - A)$$

$P(B - A) \geq 0, \quad \because$ Axiom 1

$$\therefore \quad P(A) \leq P(B).$$



□

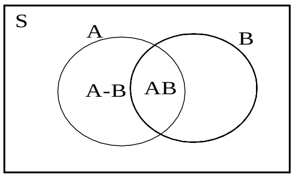
Theorem 1.3.4. *If* A *and* B *are two events, then* $P(A - B) = P(A) - P(AB)$

Proof. The event A can be divided into two mutually exclusive events $A - B$ and AB as shown in venn diagram. Thus,

$$P(A) = P(A - B) + P(AB)$$

$$P(A - B) = P(A) - P(AB)$$

$$\therefore \quad P(A) \leq P(B).$$



□

Theorem 1.3.5. *If* A *and* B *are two events, then* $P(A+B) = P(A) + P(B) - P(AB)$

Proof. The events $A + B$ can be divided into two mutually exclusive events $A - B$ and B as shown above Figure. Thus,

$$P(A + B) = P(A - B) + P(B)$$

$$= P(A) - P(AB) + P(B) \qquad \because\ (Theorem 4)$$

$$\therefore\ P(A + B) = P(A) + P(B) - P(AB).$$

If $A \cap B = \varphi$ then $P(A \cup B) = P(A + B) = P(A) + P(B)$ □

Example 1.3.1. *If two coins tossed simultaneously , Determine the probability of obtaining exactly two heads.*

***Solution**: Number of sample points* $= 2 \times 2 = 4$

$S = \{(T, T), (T, H), (H, T), (H, H)\};$ $\quad P(getting\ two\ heads) = \frac{1}{4}$

Question. 2: A Box contain 3 White, 4 Red, and 5 Black balls. A ball is drawn at randomly. Find Probability i.e., i) Red ii) Not black iii) Black or White.

Solution:

(i) Red balls = 4; White balls = 3; Black balls = 5;

$$P(Red) = \frac{\text{Ways of choosing a Red ball}}{\text{Total ways of choosing a ball}} = \frac{4}{3+4+5} = \frac{4}{12} = \frac{1}{3}$$

(ii)

$$P(Black) = \frac{5}{12}$$

$$P(Not\ a\ Black) = 1 - P(Black) = 1 - \frac{5}{12} = \frac{7}{12}$$

(iii)

$$P(White\ or\ Black) = P(B + W)$$

$$= P(B) + P(W) = \frac{5}{12} + \frac{3}{12} = \frac{2}{3}$$

Question. 3: A Bag contain 12 balls numbered from 1 to 12. If a ball is taken at random. What is the Probability having a ball with a number. Which is multiple of either 2 or 3?

Solution: Let A is an event that ball is multiples of 2

B is an event that ball is multiples of 3

$A = \{2, 4, 6, 8, 10, 12\};$ $\quad B = \{3, 6, 9, 12\};$ $\quad then\ A \cap B = \{6, 12\};$

$P(A) = \frac{6}{12};$ $\quad P(B) = \frac{1}{3};$ $\quad P(A \cap B) = \frac{2}{12} = \frac{1}{6};$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{8}{12} = \frac{2}{3}$

Hence, the required probability is $\frac{2}{3}$.

Question. 4: A coin is tossed four times in succession. Determine the probability of obtaining exactly two heads?

Solution: Sample points: $2^4 = 16$

Sample space :

0000 0001 0010 0011 0100 0101 0110 0111

1000 1001 1010 1011 1100 1101 1110 1111

$P(\text{exactly two heads}) = \frac{6}{16} = \frac{3}{8}$

Question. 5: A die is tossed find the probability of event $A = \{\text{odd number}\}$, $B = \{\text{number larger than } 3 \text{ show up}\}$, $A \cup B$ and $A \cap B$.

Solution: Sample space: $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$ $B = \{4, 5, 6\}$

$A \cup B = \{1, 3, 4, 5, 6\}$ $A \cap B = \{5\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$ $P(B) = \frac{3}{6} = \frac{1}{2}$

$P(A \cup B) = \frac{5}{6}$ $P(A \cap B) = \frac{1}{6}$

Verification: $A \cup B = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$

Question. 6: An experiment consists of rolling a single dice, two events are defined as $A = \{a\ 6\ show\ up\}$; $B = \{a\ 2\ or\ a\ 5\ show\ up\}$;
(i) Find $P(A)$ and $P(B)$ (ii) $P(C) = 1 - P(A) - P(B)$

Solution: $A = \{a\ 6\ show\ up\}$; $B = \{2, 5\}$; $P(A) = \frac{1}{6}$

$P(B) = P(2 \cup 5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(C) = 1 - P(A) - P(B) = 1 - \frac{1}{6} - \frac{2}{6} = \frac{1}{2}$

Question. 7: A pair of dice are thrown. Person A wins if sum of number showing up is six or less and one of the dice shows four. Person B wins if the sum is five or more and one of the dice shows a four. Find (a) Probability that A wins. (b) The probability that B wins. (c) The probability that both A and B wins.

Solution:

(a) Person $A \longrightarrow$ sum of number is six or less (≤ 6), but one dice 4.
$P(A) = P(A\ wins) = P(2, 4) + P(1, 4) + P(4, 2) + P(4, 1) = \frac{4}{36}$

(b) Person $B \longrightarrow$ sum of number is five or more (≥ 5), but one dice 4.

$$\begin{aligned} P(B) &= P(B\ wins) \\ &= P(4, 1) + P(4, 2) + P(4, 3) + P(4, 4) + P(4, 5) + P(4, 6) \\ &\quad + P(1, 4) + P(2, 4) + P(3, 4) + P(5, 4) + P(6, 4) \\ &= \frac{11}{36} \end{aligned}$$

Question. 8: When three dice are thrown. What is the probability that sum on three faces is less than 16.

Solution: Sample space S: $6 \times 6 \times 6 = 216$

$$\begin{aligned} P(sum < 16) &= 1 - P(sum \geq 16) \\ &= 1 - \{P(sum = 16) + P(sum = 17) + P(sum = 18)\} \\ &= 1 - \frac{6}{216} + \frac{3}{216} + \frac{1}{216} = 1 - \frac{10}{216} = \frac{206}{216} \end{aligned}$$

Question. 9: Two dice are thrown. Determine,

1. The probability that sum on the dice is *seven*, i.e., $P(A) = P(7)$.
2. The probability of getting sum *ten* or *eleven*, i.e., $P(B)$.
3. The probability of getting the sum between 8 to 11,
   i.e., $P(C) = P(8 < sum \leq 11)$.
4. The probability of getting sum greater than 10, i.e., $P(D)$.

5. $P\{(8 < sum \le 11) \cup (10 < sum)\}$.

6. $P\{(8 < sum \le 11) \cap (10 < sum)\}$.

7. $P\{(sum \ge 10)\}$.

8. A die will shows a 2 and the other will shows 3 or larger.

9. $P(10 \le sum\ and\ sum \le 4)$

10. Let X and Y denote the numbers are the first and second die respectively. Find $(i)\ P[X = Y]$ $(ii)\ P[(X + Y) = 8]$ $(iii) P[(X + Y) \ge 8]$ $(iv)\ P(7\ or\ 11)$ (v) X be the event that Y is larger than 3. Find $X,\ P(X)$.

Solution: Number of sample points in sample space = $6 \times 6 = 36$

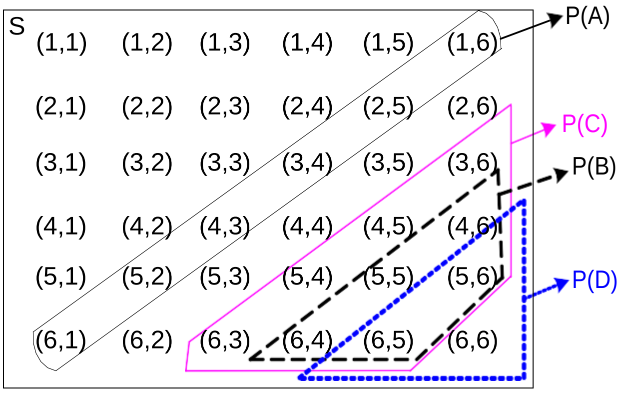


Fig. 1.2 Dice Sample Space

1. $P(A) = P(7) = \frac{6}{36} = \frac{1}{6}$

2. $P(B) = P(10\ or\ 11) = \frac{5}{36}$

3. $P(C) = P(8 < sun \le 11) = \frac{14}{36}$

4. $P(D) = P(sum > 10) = \frac{3}{36} = \frac{1}{12}$

5. $P\{(8 < sum \le 11) \cup (10 < sum)\} = \frac{10}{36} = \frac{5}{18}$

6. $P\{(8 < sum \le 11) \cap (10 < sum)\} = \frac{2}{36} = \frac{1}{18}$

7. $P(sum \ge 10) = \frac{6}{36} = \frac{1}{6}$

8. $P(2\ and \ge 3) = \{(2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (4, 2), (5, 2), (6, 2)\}$

$= 8\frac{1}{36} = \frac{2}{9}$

9. $P(10 \le sum\ and\ sum \le 4) = \frac{12}{36} = \frac{1}{3}$

10. (i) $P[X = Y] = \frac{6}{36} = \frac{1}{6}$

(ii) $P[(X + Y) = 8] = \frac{5}{36}$

(iii) $P[(X+Y) \geq 8] = \frac{15}{36}$

(iv) $P(7 \; or \; 11) = P(7) + P(11) - P(7 \cap 11) = \frac{6}{36} + \frac{2}{36} - 0 = \frac{8}{36} = \frac{4}{17}$

(v) $X = \{(x, y) : x \in N, y \in N, 1 \leq x \leq 6, 4 \leq y \leq 6\}$ and

$$P(X) = \frac{x}{S} = \frac{18}{36} = \frac{1}{2}$$

1.3.5 Probability: Playing Cards

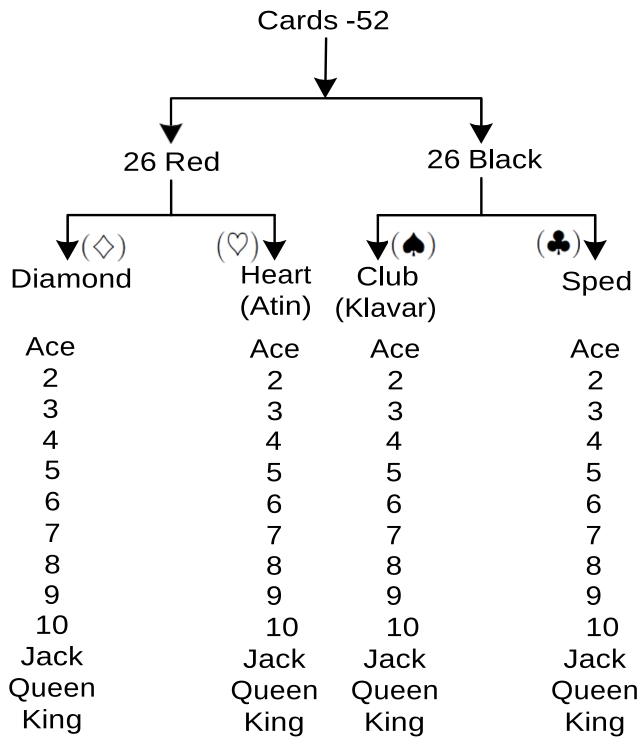


Fig. 1.3 Playing Cards table

The primary deck of 52 playing cards in use today and includes thirteen ranks of each of the four French suits, diamonds (♦), spades (♠), hearts (♥) and clubs (♣), with reversible Rouennais “court” or face cards (some modern face card designs, however, have done away with the traditional reversible figures).

Each suit includes an ace, depicting a single symbol of its suit; a king, queen, and jack, each depicted with a symbol of its suit; and ranks two through ten, with each card depicting that many symbols (*pips*) of its suit.

Two (sometimes one or four) Jokers, often distinguishable with one being more colorful than the other, are included in commercial decks but many games require one or both to be removed before play . . . A deck often comes with two Joker Cards that do not usually have hearts, diamonds, clubs or spades, because they can be any card in certain games. In most card games, however, they are not used.

Question. 10: A card is drawn at random from an ordinary deck of 52 playing cards. Find the probability of its being (a) an ace; (b) a six or a Heart; (c) neither a nine nor a spade; (d) either red or a king; (e) 5 or smaller; (f) red 10.

Solution:

(a) $P(Ace) = \dfrac{4\,Aces}{52\ Cards} = \dfrac{1}{13}$

(b) $P(6+H) = P(6) + P(H) - P(6H) = \dfrac{4}{52} + \dfrac{13}{52} - \dfrac{1}{52} = \dfrac{4}{13}$

(c)

$$
\begin{aligned}
P(\overline{9S}) &= P(\overline{9+S}) \\
&= 1 - P(9+S) \\
&= 1 - [P(9) + P(S) - P(9S)] \\
&= 1 - \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{9}{13}
\end{aligned}
$$

(d)

$$
\begin{aligned}
P(R \cup K) &= P(R) + P(K) - P(RK) \\
&= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{7}{13}
\end{aligned}
$$

(e)

$$
\begin{aligned}
P(cards \le 5) &= \frac{4\ fives + 4\ fours + 4\ threes + 4\ twos}{52\ cars} \\
&= \frac{4+4+4+4}{52} = \frac{16}{52} = \frac{4}{13}
\end{aligned}
$$

(f) $P(red\ 10) = \dfrac{[1\ ten\ of\ heart + 1\ ten\ of\ diamond]}{52\ cards} = \dfrac{2}{52}$

Question. 11 In an experiment of drawing a card from a pack the event of getting a spade is denoted by A, getting a pictured card (king, queen or jack) is denoted by B. Find the probability of $A, B, A \cap B, A \cup B$.

Solution:

$$P(A) = \frac{13}{52}; \qquad P(B) = \frac{4 \times 3}{52} = \frac{12}{52}; \qquad P(A \cap B) = \frac{3}{52};$$

$$P(A \cup B) = P(A) + P(B) - P(AB) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}$$

1.4 Conditional, Joint Probabilities and Independent events

1.4.1 Conditional Probability

Let us consider 'A' and 'B' are two events of a random experiment. conditional probability is defined as

$$P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{P(A \cap B)}{P(A)}; \quad P(A) \quad 0; \tag{1.4}$$

Here $P(A)$ is called elementary probability; $P(AB)$ is joint probability; $P\left(\frac{B}{A}\right)$ is conditional probability. i.e., the probability of B given that event A has already occurred.

1.4.2 Joint probability

The joint probability of two events may be expressed as the product of the conditional probability of one event given the other, and elementary probability of the other.

$P(AB) = P\left(\frac{B}{A}\right).P(A) = P\left(\frac{A}{B}\right).P(B)$ (chin rule or multiplication rule)

If occurrence of event A does not effect the occurrence of event B,

then $P\left(\frac{B}{A}\right) = P(B)$,

then, it is called A and B are *statistically independent events*.

$$P(A \cap B) \; or \; P(AB) = P(A)P(B) \tag{1.5}$$

$P(A \cap B) = P(AB) = \varphi$ is a *null event*, then A and B are called *mutually exclusive event.*

Question. 12: In a box there are 100 resistors having resistance and tolerance as given in Table.

Resistance	Tolerance		
	5%	10%	Total
22Ω	10	14	24
47Ω	28	16	44
100Ω	24	8	32
Total	62	38	100

1. $P(A), P(B), P(C)$
2. $P(AB), P(BA), P(CA)$
3. $P\left(\frac{A}{B}\right), P\left(\frac{A}{C}\right), P\left(\frac{B}{C}\right)$
4. Is A, B, C are independent.

Let a resistor be selected from the box and assume each resistor has the same likely-hood of being chosen. Define three events are A as draw a 47Ω resister, B as draw 5% tolerance resistor and C as draw 100Ω resistor. Now find elementary probabilities, joint and conditional probability.

Solution:

1. Probability of getting 67Ω resistor: $P(A) = \frac{44}{100}$
   Probability of getting 5% tolerance: $P(B) = \frac{62}{100}$
   Probability of getting 100Ω resistor: $P(C) = \frac{32}{100}$
2. Probability of the resistor building 47Ω and 5% tolerance: $P(AB) = \frac{28}{100}$
   Probability of the resistor building 100Ω and 5% tolerance: $P(BC) = \frac{24}{100}$
   Probability of the resistor building 47Ω and 100Ω resistor: $P(AC) = \varphi = 0$

3. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(A)} = \frac{28}{100} \times \frac{100}{62} = \frac{14}{31}$

$P\left(\frac{A}{C}\right) = \frac{P(AC)}{P(C)} = 0$

$P\left(\frac{B}{C}\right) = \frac{P(BC)}{P(C)} = \frac{24}{100} \times \frac{100}{32} = \frac{3}{4}$

4. $P(AB) = P(A) \times P(B)$

$P(AB) = \frac{28}{100} = \frac{7}{25}$

$P(A) \times P(B) = \frac{44}{100} \times \frac{62}{100} = \frac{11}{50} \times \frac{31}{25}$

So, $P(AB)$ and $P(A) \times P(B)$ values are unequal. Therefore they are dependent.

1.4.3 Properties of conditional probability

1. For any two events A and B in sample space. If $B \subset A$ then $P\left(\frac{A}{B}\right) = 1$

Proof:

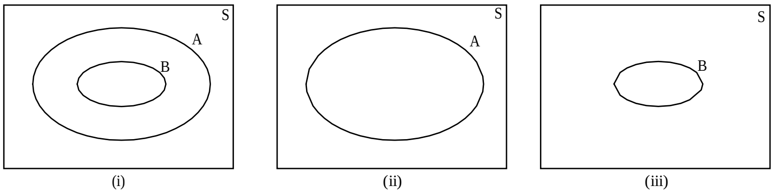


Fig. 1.4

$P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)} = \frac{P(B)}{P(B)} = 1$ $\because$ From Fig. 1.4 (i), $P(AB) = P(B)$

2. If $B \subset A$ then $P\left(\frac{B}{A}\right) = \frac{P(B)}{P(A)}$

Proof: $P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{P(B)}{P(A)}$

3. $P\left(\frac{A}{B}\right) \geq 0$, it is non-negative.

Proof: $P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)}$ then $P(AB) \geq 0$;

$P(B) \neq 0$. So, $P\left(\frac{A}{B}\right) \geq 0$

4. If two events A and B are in the sample space S then,

- $P\left(\frac{S}{A}\right) = P\left(\frac{S}{B}\right) = 1$

  Proof: From Fig. 1.4 (ii) and (iii) then,

  $P\left(\frac{S}{A}\right) = P\left(\frac{SA}{A}\right) = \frac{P(A)}{P(A)} = 1$; $P(A)$ 0

  Similarly, $P\left(\frac{S}{B}\right) = \frac{P(SB)}{P(B)} = \frac{P(B)}{P(B)} = 1$; $P(B)$ 0

- $P\left(\frac{A}{S}\right) = P(A)$ and $P\left(\frac{B}{S}\right) = P(B)$

  Proof: $P\left(\frac{A}{S}\right) = \frac{P(AS)}{P(S)} = \frac{P(A)}{1} = P(A)$

  similarly, $P\left(\frac{B}{S}\right) = \frac{P(BS)}{P(S)} = \frac{P(B)}{1} = P(B)$

1.4.4 Joint Properties and Independent events

1.4.4.1 Joint Properties

$P(AB) = P(A).P(\frac{B}{A})$

1.4.4.2 Independent

If $P(B/A) = P(B)\ then\ P(AB) = P(A)P(B)$

If $P(AB) = 0$, then it is mutually exclusive events.

If the random experiment is consisting on 'n' events, i.e., $A_1, A_2, \ldots, A_n$. If n events are independent then $P(A_1, A_2, \ldots, A_n) = P(A_1)P(A_2)\ldots P(A_n)$ is *probability multiplication theorem.*

Theorem 1.4.6. *If events A and B are independent then $\bar{A}$ and B, A and $\bar{B}$, $\bar{A}$ and $\bar{B}$ are also independent.*

1. $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$
2. $P(\bar{A} \cap B) = P(\bar{A})P(B)$
3. $P(A \cap \bar{B}) = P(A)P(\bar{B})$

Proof. 1.

$$
\begin{aligned}
P(\bar{A} \cap \bar{B}) &= P(\overline{\bar{\bar{A}} \cup \bar{\bar{B}}}) = P(\overline{A \cup B}) \\
&= 1 - P(A \cup B) \\
&= 1 - \left[P(A) + P(B) - P(AB)\right] \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= \left[1 - P(A)\right] - P(B)\left[1 - P(A)\right] \\
&= \left[1 - P(A)\right]\left[1 - P(B)\right] \\
&= P(A)P(\bar{B})
\end{aligned}
$$

2.

$$
\begin{aligned}
P(\bar{A} \cap B) &= P(\overline{\bar{\bar{A}} \cup \bar{B}}) = P(\overline{A \cup \bar{B}}) \\
&= 1 - P(A \cup \bar{B}) \\
&= 1 - \left[P(A) + P(\bar{B}) - P(A \cap \bar{B})\right] \\
&= 1 - P(A) - P(\bar{B}) + P(A)P(\bar{B}) \\
&= P(\bar{A}) - P(\bar{B})\left[1 - P(A)\right] \\
&= P(\bar{A})\left[1 - P(\bar{B})\right] \\
&= P(\bar{A})P(B)
\end{aligned}
$$

3.

$$P(A \cap \bar{B}) = P(\overline{\bar{A} \cup \bar{\bar{B}}}) = P(\overline{\bar{A} \cup B})$$
$$= 1 - P(\bar{A} \cup B)$$
$$= 1 - P(\bar{A}) - P(B) + P(\bar{A})P(B)$$
$$= P(A) - P(B)\left[1 - P(\bar{A})\right]$$
$$= P(A) - P(B)P(A)$$
$$= P(A)\left[1 - P(B)\right]$$
$$= P(A)P(B)$$

□

Question. 13: One card is selected from ordinary 52 cards and events defined as event A as select a king, event B as Jack and Queen and event C as select Heart. Determine whether A, B and C are independent.

Solution:

- Event A select a King: $P(A) = P(King) = \frac{4}{52}$
- Event B select a Jack and Queen: $P(B) = P(J \cup Q) = P(J) + P(Q) = \frac{8}{52}$
- Event A select a Heart: $P(C) = P(Heart) = \frac{13}{52}$
- $P(AB) = P(King\ and\ Jack\ or\ Queen) = 0$
- $P(BC) = P(Jack\ or\ Queen\ and\ Heart) = \frac{2}{52}$
- $P(CA) = P(Heart, King) = \frac{1}{52}$
- $P(AB) = P(A).P(B) \implies 0 \neq \frac{4}{52}.\frac{8}{52}$

  So, A and B are dependent.
- $P(BC) = P(B).P(C) \implies \frac{2}{52} = \frac{8}{52}.\frac{13}{52} \implies \frac{2}{52} = \frac{2}{52}$

  So, B and C are independent.
- $P(AC) = P(A).P(C) \implies \frac{1}{52} = \frac{4}{52}.\frac{13}{52} \implies \frac{1}{52} = \frac{1}{52}$

  Hence, A and C are independent.

  ∴ $P(AB) = 0$ the A and B are Mutually exclusive events.

Question. 14: Find the probability of drawing first card is diamond and second card is heart (first card is not replaced).

Solution: $P(DH) = P(D).P\left(\frac{H}{D}\right) = \frac{13}{52} \times \frac{13}{51}$

Question. 15: What is the probability that a six is obtained on one the dice in a throw two dice, given that the sum is 7.

Solution: Let A be the event getting sum is 7 and B the event, 6 appears on any one of the dice.

$A = \{(2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (1, 6)\}$ $\quad So,\ P(A) = \frac{6}{36} \quad P(A \cap B) = \frac{2}{36}$

$$P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{2}{6} = \frac{1}{3}$$

Question. 16: A pair of dice thrown, find the probability that the sum is 10 or greater, if

(i) 5 appears on the first dice

(ii) 5 appears on the at least one of the dice.

Solution:

(i) Let A be the event that 5 appears on the first dice then

$A = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

Total sample space $S = 6 \times 6 = 36$, therefore $P(A) = \frac{6}{36}$

Let B the event that sum is 10 or greater, $P(A \cap B) = \frac{2}{36}$

$$\therefore P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}$$

(ii) Let C be the event that 5 appears on atleast one of the dice, then

$C = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5)\}$

$P(A) = \frac{11}{36}$

$P(AC) = \frac{3}{36}$

$$P\left(\frac{C}{A}\right) = \frac{P(CA)}{P(A)} = \frac{\frac{3}{36}}{\frac{11}{36}} = \frac{3}{11}$$

1.5 Total Probability

Let us consider a random experiment whose sample space 'S which consisting of 'n' mutually exclusive events. i.e., B_n the probability of elementary event A in terms of all mutually exclusive events called total probability and it is written as $P(A)$.

$$P(A) = \sum_{m=1}^{n} P(B_n)\, P\left(\frac{A}{B_n}\right) \tag{1.6}$$

Proof. Let us consider a simple space 'S' as shown in Fig. 1.5 Here B_n is exclusive events. i.e., $B_m \cap B_n = \varphi;\ m \quad n = 1, 2, 3, \ldots N$

$$S = \cup_{n=1}^{N} B_n = \sum_{n=1}^{N} B_n$$

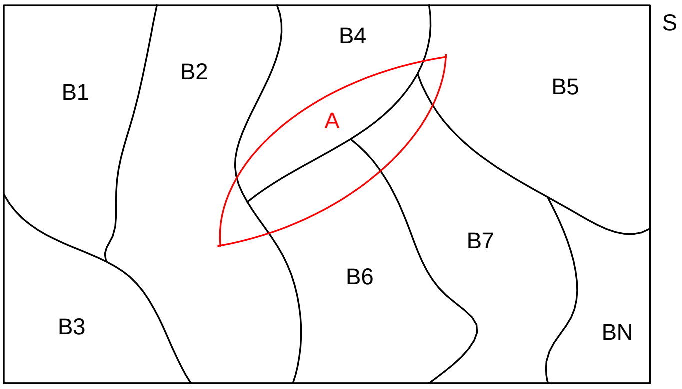


Fig. 1.5 Total probability

From Fig. event 'A' in terms of sample space 'S' can be written as,

$$A = A \cap S = S \cap A = A \cap \sum_{n=1}^{N} B_n$$

$$= \sum_{n=1}^{N} A \cap B_n$$

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n); \qquad (\because \; Axiom \; 3)$$

$$P\left(\frac{A}{B_n}\right) = \frac{P(A \cap B_n)}{P(B_n)}; \; P(B_n) \quad 0;$$

Given that N mutually exclusive events B_n, $n = 1, 2, 3, \ldots N$, whose union equals the sample space S on the sample space. The probability of any event A, $P(A)$ can be written in terms of conditional probability as,

$$P(A) = \sum_{n=1}^{N} P(B_n) P\left(\frac{A}{B_n}\right)$$

This result is known as "*Total Probability*" of event A. □

Question. 17: There are three boxes such that box one contain 3 red, 4 green, 5 blue; box two contain 4 red, 3 green, 4 blue; and box three contain 2 red, 1 green, 4 blue; these boxes are selected randomly with equal probability and then one ball is drawn from the selected box. Find the probability that the drawn ball is red.

Solution:

$$P(Box_1) = P(Box_2) = P(Box_3) = \frac{1}{3}$$

$$P(Red) = \sum_{n=1}^{3} P(Box_n)P\left(\frac{Red}{Box_n}\right)$$

$$= P(Box_1)P\left(\frac{Red}{Box_1}\right) + P(Box_2)P\left(\frac{Red}{Box_2}\right) + P(Box_3)P\left(\frac{Red}{Box_3}\right)$$

$$= \frac{1}{3}\times\frac{3}{12} + \frac{1}{3}\times\frac{4}{9} + \frac{1}{3}\times\frac{2}{7}$$

$$= \frac{1}{3}\frac{242}{252}$$

1.6 Baye's Theorem

It is a rule of inverse probability or rule of inverse conditional probability.

Let us consider a random experiment where sample space 'S' such that consists of 'n' mutually exclusive events. Now the probability of $P\frac{B_i}{A}$ is

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P(\frac{A}{B_i})}{P(A)} \tag{1.7}$$

$$where\ P(A) = \sum_{i=1} P(B_i)P\left(\frac{A}{B_i}\right),\ then$$

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P(\frac{A}{B_i})}{\sum_{i=1}^{n} P(B_i)P(\frac{A}{B_i})} \tag{1.8}$$

Here, $P(B_i)$ is priori probabilities i.e., event before performance of experiment. $P(A)$ is total probability.

- This Baye's theorem formulae widely used in biometrics, epidemiology and communication theory.
- The term $P(\frac{A_i}{B})$ is known as the posteriori probability of an given B and $P(\frac{B}{A_i})$ is called a priori probability of B given A_i and $P(A_i)$ is the casual or a priori probability of A_i.
- In general a priori probability are estimated from past measurements or presupposed by experience while a posteriori probabilities are measured or computed from observations.
- A example of Baye's formula is the Binary Symmetric Channel (BSC) shown in next Fig. (1.6), which model for bit errors that occur in a digital communication system. For binary system, the the transmitted symbols has two outcomes $\{0, 1\}$.

Question. 18 Determine probabilities of system error and correct transmission of symbols of a binary communication channel as shown in Fig. 1.6

It consisting of transmitter that transmits one of two possible symbols 0 or 1 over a channel to receiver. The channel causes error so that symbol 1 is converted to 0 and vice-versa at the receiver. Assume the symbols 1 and 0 are selected for the transmission as 0.6 and 0.4 respectively.

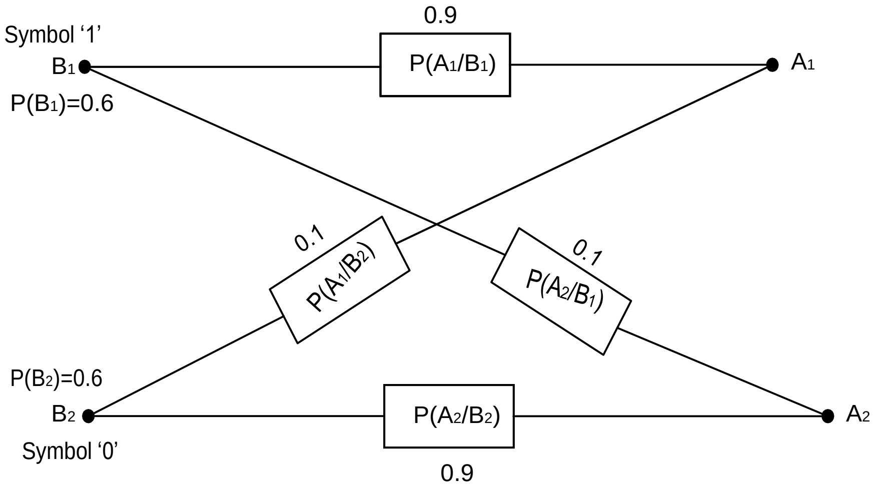


Fig. 1.6 Binary Symmetric channel

Solution:

The effect of the channel on the transmitted symbol is described by conditional probabilities.

- Let $P(B_1)$ represents the probability of transmitting symbol '1'.
- $P(B_2)$ represents the probability of transmitting symbol '0'.
- The conditional probabilities or transmitting probabilities are

$$P\frac{A_1}{B_1}, P\frac{A_2}{B_1}, P\frac{A_1}{B_2}, P\frac{A_2}{B_2}$$

- $P(A_1)$ represents probability of receiving symbol '1'
- $P(A_2)$ represents probability of receiving symbol '0'
- The reception probabilities given as '1' was transmitted to $P\frac{A_1}{B_2} = 0.1; P\frac{A_2}{B_1} = 0.9$
- Channel effect $0^{\prime}s$ in the same manner, $P\frac{A_1}{B_2} = 0.9; P\frac{A_2}{B_2} = 0.1$
- As seen in both cases, $P\frac{A_1}{B_i} + P\frac{A_2}{B_i} = 1$. Since, A_1 and A_2 are mutually exclusive and are the only receiver events possible.

$P\left(\frac{B_1}{A_1}\right)$ $\longrightarrow$ Probability of occurrence of B_1 provided A_1 as already occurred, then what is the probability that the received A_1 is due to B_1 in communication.

From the theorem of total probability,

$$P(A) = \sum_{n=1}^{N} P\left(\frac{A}{B_n}\right);$$

where $\sum_{n=1}^{N} B_n = S$ and N = total number of events in 'S'. By using above theorem we obtain probabilities of A_1 and A_2; i.e., *received symbol probabilities* are,

$$P(A_1) = P\left(\frac{A_1}{B_1}\right)P(B_1) + P\left(\frac{A_1}{B_2}\right)P(B_2)$$

$$= 0.9 \times 0.6 + 0.1 \times 0.4 = 0.54 + 0.04 = 0.58$$

$$P(A_2) = P\left(\frac{A_2}{B_1}\right)P(B_1) + P\left(\frac{A_2}{B_2}\right)P(B_2)$$

$$= 0.1 \times 0.6 + 0.9 \times 0.4 = 0.42$$

Let $P\left(\frac{B_1}{A_2}\right)$ and $P\left(\frac{B_2}{A_1}\right)$ are *probabilities of system error* then,

$$P\left(\frac{B_1}{A_2}\right) = \frac{P\left(\frac{A_2}{B_1}\right)P(B_1)}{P(A_2)} = \frac{0.1 \times 0.6}{0.42} = 0.143$$

$$P\left(\frac{B_2}{A_1}\right) = \frac{P\left(\frac{A_1}{B_2}\right)P(B_2)}{P(A_1)} = \frac{0.1 \times 0.4}{0.58} = 0.069$$

Let $P\left(\frac{A_1}{B_1}\right)$ and $P\left(\frac{A_2}{B_2}\right)$ are represents the *probability correct system transmission of symbols* and are obtained by using given Baye's theorem,

$$P\left(\frac{B_1}{A_1}\right) = \frac{P\left(\frac{A_1}{B_1}\right)P(B_1)}{P(A_1)} = \frac{0.9 \times 0.6}{0.58} = 0.931$$

$$P\left(\frac{B_2}{A_2}\right) = \frac{P\left(\frac{A_2}{B_2}\right)P(B_2)}{P(A_2)} = \frac{0.9 \times 0.4}{0.42} = 0.857$$

Problem: 19 A bag 'X' contains 3 white and 2 black balls another bag contains 2 white and 4 black balls. If one bag is selected at random and a ball is selected from it then find the probability that ball is white.

Solution: Given

$X \Longrightarrow B_1$ contains 3 White and 2 Black

$Y \Longrightarrow B_2$ contains 2 White and 4 Black

$P(Bag_1) = P(Bag_2) = \frac{1}{2}$

Total probability: $P(A) = \sum_{n=1}^{N} P(A|B_n) \cdot P(B_n)$

$$\therefore P(W) = \sum_{n=1}^{2} P(W|B_n) \cdot P(B_n)$$

$$= P(W|B_1) \cdot P(B_1) + P(W|B_2) \cdot P(B_2)$$

$$= \frac{3}{5} \times \frac{1}{2} + \frac{2}{6} \times \frac{1}{2}$$

$$= 0.46667$$

CHAPTER 2

Random Variables or Stochastic Variables

2.1 Random variable

A random variable is a real valued function defined over a sample space of a random experiment, it is called random or stochastic variable. Random variables are denoted by capital or upper case letters such as X, Y etc., and the values assumed by are denoted by lower case letters with subscripts such as x_1, x_2, y_1, y_2 etc.,

Example: Let us consider a random experiment is tossing three coins, there are eight possible outcomes of this experiment. The sample space can be written as,

S	=	$HHH,$	$HHT,$	$HTH,$	$HTT,$	$THH,$	$THT,$	$TTH,$	TTT
X	=	$x_1,$	$x_2,$	$x_3,$	$x_4,$	$x_5,$	$x_6,$	$x_7,$	x_8
Cond.	=	3	2	2	1	2	1	1	0

Here, S denotes a sample space, X denotes a random variable and the condition is number of heads.

2.1.1 Conditions for a function to be a random variable

- A random variable should be a single valued function i.e., every sample point in sample space 'S' must correspond to only one value of the random variable.

- The probability of event $a \leq X \leq b$ is equal to sum of the probabilities of all elements between a and b.

- The probability of the events $\{X = -\infty\}$ and $\{X = +\infty\}$ should be zero.

2.1.2 Types of random variable

1. Discrete random variable: If the random variable takes finite set of discrete values then it is called discrete random variable.

   Ex: In tossing a three coins, a random variable 'X' takes $0, 1, 2$ and 3 values.

2. Continuous random variable: If the variable takes infinite set of values then it is called continuous random variable.

Ex: Finding the real value between 0 to 12 in a random experiment, sample space S will be $\{0 \leq S \leq 12\}$.

3. Mixed random variable: If the random variable takes both discrete and continuous values then it is called mixed random variable.
   Ex: Let us consider a random experiment in which the temperature is measured by selecting thermometer randomly, then selection of thermometer takes finite values it is called discrete random variable and measuring the temperature takes continuous value and it is called continuous random variable. Combination of these two is called mixed random variable.

2.2 Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of a Discrete Random Variable

The Probability of a random variable is called probability density function (PDF) or probability mass function (PMF) of a discrete random variable. It is represented as $P\ (X = x) = f_X(x);\ \{1 \leq X \leq 1\}$. The cumulative addition probability density function from $X = -\infty\ to\ \infty\}$ is called probability distribution or cumulative distribution function (CDF) and it is denoted by $F_X(x) = P\{-\infty \leq X \leq \infty\}$.

Question. 1: Find the PDF and CDF of a random experiment in which three coins are tossed and condition to get random variable is getting head.

Solution:

Sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Random variable $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

No.of Heads (Condition) = $\{3\ 2\ 2\ 1\ 2\ 1\ 1\ 0\}$

Apply the condition to random variable X, getting head $X = \{01234\}$ The probability density function (PDF) is the probability of random variable.

$$P(X = 0) = P(x_1) = \frac{1}{8}$$
$$P(X = 1) = P(x_2) + P(x_3) + P(x_5) = \frac{3}{8}$$
$$P(X = 2) = P(x_4) + P(x_6) + P(x_7) = \frac{3}{8}$$
$$P(X = 3) = P(x_8) = \frac{1}{8}$$

The probability density function (PDF) is given by

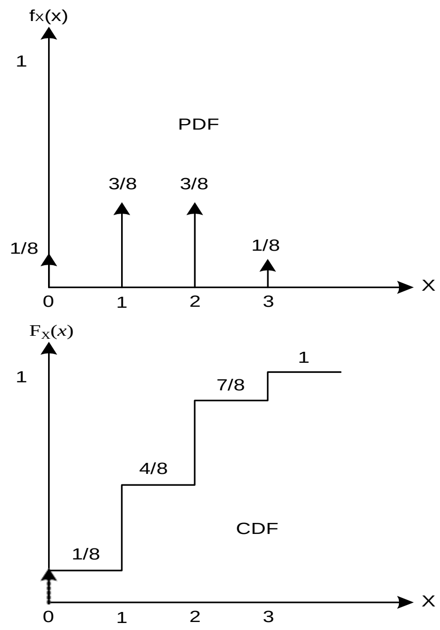
X	0	1	2	3
$f_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The expression for probability density function is

$$f_X(x) = \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3)$$

The expression for CDF function is

$$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1) + \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3)$$



Question. 2: Two dies are rolls, find PDF and CDF of a random variable 'X' which is getting sum on two dies.

Solution: The sample space 'S'

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Condition to get random variable getting sum of two dice.

$X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$P(X = 2) = \frac{1}{36}$ $\quad P(X = 3) = \frac{2}{36}$ $\quad P(X = 4) = \frac{3}{36}$

$P(X = 5) = \frac{4}{36}$ $\quad P(X = 6) = \frac{5}{36}$ $\quad P(X = 7) = \frac{6}{36}$

$P(X = 8) = \frac{5}{36}$ $\quad P(X = 9) = \frac{4}{36}$ $\quad P(X = 10) = \frac{3}{36}$

$P(X = 11) = \frac{2}{36}$ $\quad P(X = 12) = \frac{1}{36}$

The expression for probability density function is

$$f_X(x) = \frac{1}{36}\delta(x-2) + \frac{1}{36}\delta(x-3) + \frac{1}{36}\delta(x-4) + \frac{1}{36}\delta(x-5) + \frac{1}{36}\delta(x-6)$$
$$+\frac{1}{36}\delta(x-7) + \frac{1}{36}\delta(x-8) + \frac{1}{36}\delta(x-9) + \frac{1}{36}\delta(x-10)$$
$$+\frac{1}{36}\delta(x-11) + \frac{1}{36}\delta(x-12)$$

The expression for CDF function is

$$F_X(x) = \frac{1}{36}u(x-2) + \frac{2}{36}u(x-3) + \frac{3}{36}u(x-4) + \frac{4}{36}u(x-5) + \frac{5}{36}\delta(x-6)$$
$$+\frac{6}{36}u(x-7) + \frac{5}{36}u(x-8) + \frac{4}{36}u(x-9) + \frac{3}{36}u(x-10)$$
$$+\frac{2}{36}u(x-11) + \frac{1}{36}u(x-12)$$

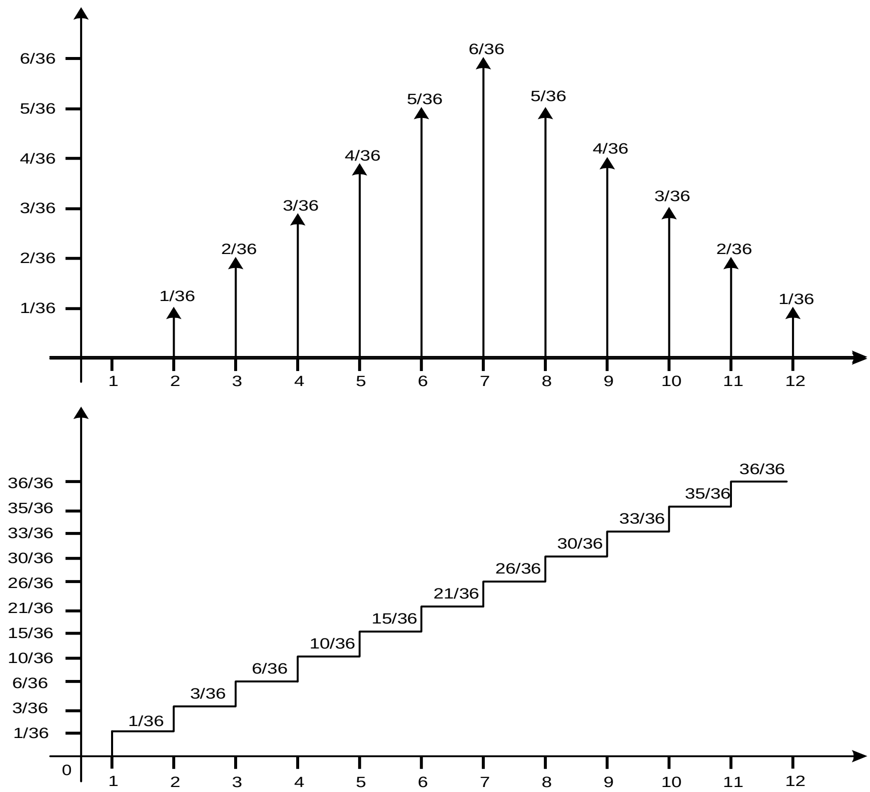


Fig. 2.1 Two dice PDF and CDF plot

Question. 3: The probability density function of a discrete random variable is given by

$X = x_i$	-1	0	1	2	3	4	5	6	7
$f_X(x)$	K	$2K$	$3K$	K	$4K$	$3K$	$2K$	$4K$	K

Find (i) K (ii) $P\{X \le 2\}$ (iii) $P\{X > 4\}$ (iv) $P\{1 < X \le 4\}$ and (v) PDF and CDF.

Solution: Total probability = 1;

(i)

$$K + 2K + 3K + K + 4K + 3K + 2K + 4K + K = 1$$
$$21K = 1$$
$$K = \frac{1}{21}$$

(ii)

$$P\{X \le 2\} = P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$$
$$= K + 2K + 3K + K$$
$$= \frac{7}{21} \qquad \because K = \frac{1}{21}$$

(iii)

$$P\{X > 2\} = P(X = 5) + P(X = 6) + P(X = 7)$$
$$= 2K + 4K + K$$
$$= \frac{7}{21} \qquad \because K = \frac{1}{21}$$

(iv)

$$P\{1 < X \le 4\} = P(X = 5) + P(X = 6) + P(X = 7)$$
$$= K + 4K + 3K$$
$$= \frac{8}{21} \qquad \because K = \frac{1}{21}$$

The expression for probability density function is

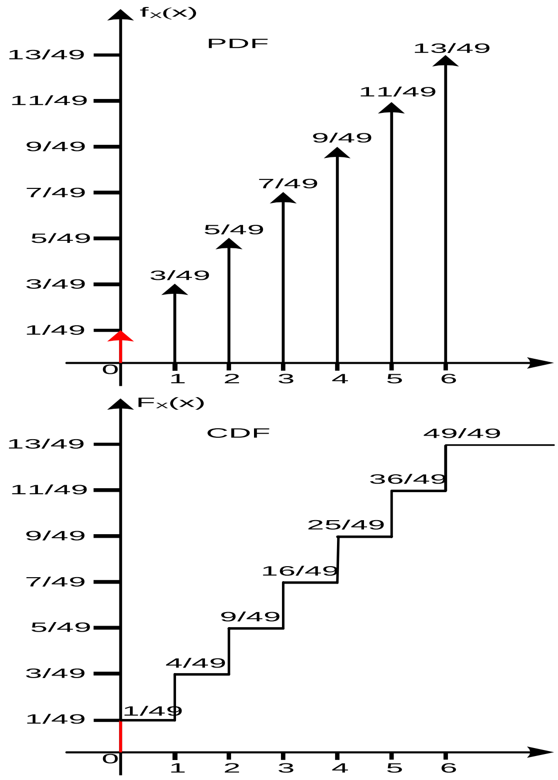
$$f_X(x) = \frac{1}{21}\delta(x+1) + \frac{2}{21}\delta(x) + \frac{3}{21}\delta(x-1) + \frac{1}{21}\delta(x-2) + \frac{4}{21}\delta(x-3) + \frac{3}{21}\delta(x-4) + \frac{2}{21}\delta(x-5) + \frac{4}{21}\delta(x-6) + \frac{1}{21}\delta(x-7)$$

The expression for CDF function is

$$F_X(x) = \frac{1}{21}u(x+1) + \frac{2}{21}u(x) + \frac{3}{21}u(x-1) + \frac{1}{21}u(x-2) + \frac{4}{21}u(x-3) + \frac{3}{21}u(x-4) + \frac{2}{21}u(x-5) + \frac{4}{21}u(x-6) + \frac{1}{21}u(x-7)$$

2.3 PDF and CDF of Continuous Random Variable

Let $f_X(x)$ and $F_X(x)$ are the PDF and CDF of a continuous random variable X.



2.3.1 Properties of PDF

- It is a *non-negative* function.
- The area under the PDF curve is unity. i.e., $\int_{x=-\infty}^{\infty} f_X(x)dx = 1.$
- The probability of a random variable between intervals a and b can be written as

$$P\{a \leq X \leq b\} = P\{a < X \leq b\} = P\{a \leq X < b\} = P\{a < X < b\}$$
$$= \int_{x=a}^{b} f_X(x)dx$$

The accumulative distribution function (CDF) of random variable X in terms of PDF can be written as

$$F_X(x) = P\{-\infty \leq X \leq x\} = P\{X \leq x\} = \int_{u=-\infty}^{x} f_X(u)du$$

2.3.2 Properties of CDF

- $F_X(x)$ has minimum value is zero at $-\infty$ and maximum value at $+\infty$ is one. i.e., $F_X(-\infty) = 0; \quad F_X(+\infty) = 1;$
- $F_X(x)$ lies between 0 to 1. i.e., $0 \leq F_X(x) \leq 1$:
- $F_X(x)$ is non decreasing function. i.e., $F_X(x_1) \leq F_X(x_2)$

Proof. Let X be the random variable, which takes the variable from $-\infty$ to $+\infty$.

$$F_X(x_2) = P(X \le x_2)$$
$$= P\Big[(-\infty \le X < x_1) \cup (x_1 \le X \le x_2)\Big]$$
$$= P\Big[(-\infty \le X < x_1) + (x_1 \le X \le x_2)\Big] \qquad \because \textit{mutually exclusive}$$
$$= P(x_1) + P(x_1 \le X \le x_2)$$

So, $F_X(x_2) > F_X(x_1)$; if $x_2 > x_1$. □

- The cumulative distribution function between x_1 and x_2 can be written as

  $P\{x_1 \le X \le x_2\} = F_X(x_2) - F_X(x_1); \quad x2 > x1$

  Proof. Let us consider X be the random variable which takes $-\infty$ to $+\infty$.

  $P\{-\infty \le X < x_1\} + P\{x_1 \le X \le x_2\} + P\{x_2 \le X \le \infty\} = 1$

  $F_X(x_1) + P\{x_1 \le X \le x_2\} + 1 - F_X(x_2) = 1$

  $\therefore \boxed{P\{-\infty \le X < x_1\} = F_X(x_2) - F_X(x_1)}$ □

- In terms of PDF and CDF can be written as $f_X(x) = \dfrac{d}{dx}F_X(x)$

Question. 4: The PDF of a continuous random variable is given by

$$f_X(x) = \begin{cases} C(x-1); & 1 \le x \le 4 \\ 0; & \text{else where} \end{cases}$$

(i) Find the value of constant 'C' (ii) Find $P\{2 \le X \le 3\}$ (iii) Plot $f_X(x)$ and $F_X(x)$

Solution:

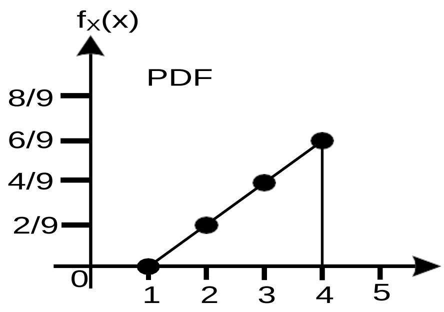
(i) Area under PDF from $-\infty$ to $+\infty$ is unity. So,

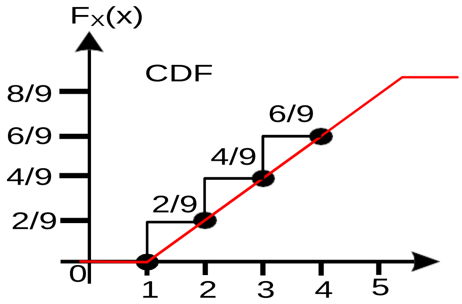
$$\int_{x=-\infty}^{\infty} f_X(x)dx = 1$$
$$\int_{x=1}^{4} C(x-1)dx = 1$$
$$C\left[\frac{x^2}{2} - x\right]_{x=1}^{4} = 1$$
$$C\left[\frac{9}{2}\right] = 1$$
$$\therefore \boxed{C = \frac{2}{9}}$$

$$f_X(x) = \begin{cases} \frac{2}{9}(x-1); & 1 \le x \le 4 \\ 0; & \text{else where} \end{cases}$$

(ii)

$$P\{2 \le X \le 3\} = \int_{x=2}^{3} \frac{2}{9}(x-1)dx$$
$$= \frac{2}{9}\left[\frac{x^2}{2} - x\right]_{x=2}^{3} = \frac{2}{9}\left[\frac{3}{2}\right] = \frac{1}{3}$$
$$\therefore \boxed{P\{2 \le X \le 3\} = \frac{1}{3}}$$





(iii) To find $F_X(x)$, we have three intervals. From Fig. (a). $F_X(x)$ for $-\infty \leq x \leq 1$ (b). $F_X(x)$ for $1 \leq x \leq 4$ (c). $F_X(x)$ for $4 \leq x \leq +\infty$.

(a). $F_X(x)$ for $-\infty \leq x \leq 1$

$$F_X(x) = P\{-\infty \leq X \leq 1\}$$
$$= \int_{u=-\infty}^{x} f_X(u)du = 0$$

$$\therefore \quad \boxed{F_X(x) = 0; \qquad -\infty \leq x \leq 1}$$

(b). $F_X(x)$ for $1 \leq x \leq 4$

$$F_X(x) = P\{-\infty \leq X \leq 1\} + P\{1 \leq X \leq 4\}$$
$$= \int_{x=-\infty}^{1} f_X(x)dx + \int_{u=1}^{x} f_X(u)du$$
$$= 0 + \int_{u=1}^{x} \frac{2}{9}(u-1)du$$
$$= \frac{2}{9}\left[\frac{u^2}{2} - u\right]_1^x$$
$$= \frac{(x-1)^2}{9}$$

$$\therefore \quad \boxed{F_X(x) = \frac{(x-1)^2}{9}; \qquad 1 \leq x \leq 4}$$

(c). $F_X(x)$ for $4 \leq x \leq +\infty$

$$F_X(x) = P\{-\infty \le X \le 1\} + P\{1 \le X \le 4\} + P\{4 \le X \le +\infty\}$$
$$= \int_{x=-\infty}^{1} f_X(x)dx + \int_{x=1}^{4} f_X(x)dx + \int_{u=4}^{\infty} f_X(u)du$$
$$= 0 + \int_{x=1}^{4} \frac{(x-1)^2}{9} + 0$$
$$= \frac{2}{9}\left[\frac{x^2}{2} - x\right]_1^4$$
$$= \frac{2}{9}\left[\frac{9}{2}\right] = 1$$

$$\therefore \quad \boxed{F_X(x) = 1; \quad 4 \le x \le +\infty}$$

$$F_X(x) = \begin{cases} 0; & -\infty \le x \le 1 \\ \dfrac{(x-1)^2}{9}; & 1 \le x \le 4 \\ 1; & x > 4 \end{cases}$$

Question. 5: The probability density function of a random variable 'X' is given by

$$f_X(x) = \begin{cases} \dfrac{1}{2a}; & -a \le x \le a \\ 0; & \text{else where} \end{cases}$$

(i) $P\left\{-\frac{a}{2} \le X \le \frac{a}{2}\right\}$ (ii) Plot $f_X(x)$ and $F_X(x)$

Solution:

(a) $$P\left\{-\frac{a}{2} \le X \le \frac{a}{2}\right\} = \int_{x=-a/2}^{a/2} \frac{1}{2a}\, dx$$
$$= \frac{1}{2a}\left[x\right]_{-a/2}^{a/2}$$
$$= \frac{1}{2a}\left[\frac{a}{2} + \frac{a}{2}\right] = \frac{1}{2}$$

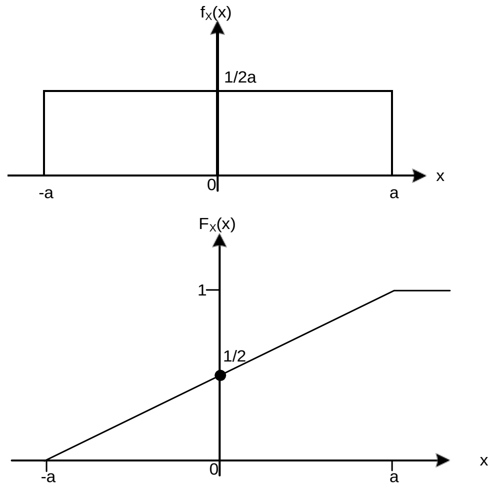
$$\therefore \quad \boxed{P\left\{-\frac{a}{2} \le X \le \frac{a}{2}\right\} = \frac{1}{2}}$$

(b) The PDF function $f_X(x)$ is shown in Fig. To find $FX(x)$, we have three intervals. i.e., (i). $F_X(x)$ for $-\infty \le x \le -a$ (ii). $F_X(x)$ for $-a \le x \le a$ (iii). $F_X(x)$ for $a \le x \le +\infty$.

(i). $F_X(x)$ for $-\infty \le x \le -a$

$$F_X(x) = P\{-\infty \le X \le -a\} = \int_{u=-\infty}^{x} f_X(u)du = 0$$

$$\therefore \quad \boxed{F_X(x) = 0; \quad -\infty \le x \le -a}$$



(ii). $F_X(x)$ for $-a \le x \le a$

$$F_X(x) = P\{-\infty \le X \le -a\} + P\{-a \le X \le a\}$$
$$= \int_{x=-\infty}^{-a} f_X(x)dx + \int_{u=-a}^{x} f_X(u)du$$
$$= 0 + \int_{u=-a}^{x} \frac{1}{2a} du = \frac{1}{2a}\left[u\right]_{-a}^{x} = \frac{1}{2a}\left(x+a\right)$$

$\therefore$ $\boxed{F_X(x) = \frac{1}{2a}\left(x+a\right) \; ; \qquad -a \le x \le a}$

(c). $F_X(x)$ for $a \le x \le +\infty$

$$F_X(x) = P\{-\infty \le X \le -a\} + P\{-a \le X \le\} + P\{a \le X \le +\infty\}$$
$$= \int_{x=-\infty}^{1} f_X(x)dx + \int_{x=-a}^{a} f_X(x)dx + \int_{u=a}^{x} f_X(u)du$$
$$= 0 + \int_{x=-a}^{a} \frac{1}{2a} dx + 0 = \frac{1}{2a}\Big[x\Big]_{-a}^{a} = \frac{1}{2a}\Big[a+a\Big] = 1$$

$\therefore$ $\boxed{F_X(x) = 1; \qquad a \le x \le +\infty}$

$$F_X(x) = \begin{cases} 0; & -\infty \le x \le -a \\ \frac{1}{2a}\left(x+a\right) \; ; & -a \le x \le a \\ 1; & a \le x \le +\infty \end{cases}$$

Question. 6: The PDF of a continuous random variable is given by

$$f_X(x) = \begin{cases} \frac{bx}{a} + b; & -a \le x \le 0 \\ -\frac{bx}{a} + b; & 0 \le x \le a \end{cases}$$

where a and b are constants. (i) Find the relation between a and b. (ii) Plot PDF and

CDF.

Solution: We know that $\int_{x=-\infty}^{\infty} f_X(x)dx = 1$

(a)

$$\int_{x=-a}^{0}\left(\frac{bx}{a}+b\right)dx+\int_{x=0}^{a}\left(-\frac{bx}{a}+b\right)dx=1$$

$$\left[\frac{b}{a}\frac{x^2}{2}+bx\right]_{-a}^{0}+\left[\frac{-b}{a}\frac{x^2}{2}+bx\right]_{0}^{a}=1$$

$$\left[0-\left(\frac{b}{a}\frac{(-a)^2}{2}+b(-a)\right)\right]+\left[\left(\frac{-b}{a}\frac{(a)^2}{2}+b(a)\right)-0\right]=1$$

$$-\left(\frac{b}{\not a}\frac{a^{\not 2}}{2}-ab\right)+\left(\frac{-b}{\not a}\frac{a^{\not 2}}{2}+ab\right)=1$$

$$\frac{-ab}{2}+ab\frac{-ab}{2}+ab=1$$

$$-ab+2ab=1$$

$$ab=1$$

$$\therefore \quad \boxed{a=\frac{1}{b}}$$

(b) From the graph of $f_X(x)$: $F_X(x)=\int_{u=-\infty}^{x} f_X(u)du$

(i) $F_X(x)$ for the interval $-\infty \le x \le -a$

$$\therefore \quad \boxed{F_X(x)=0; \qquad -\infty \le x \le -a}$$

(ii) $F_X(x)$ for the interval $-a \le x \le 0$

$$F_X(x)=\int_{x=-\infty}^{-a} f_X(x)dx+\int_{u=-a}^{x} f_X(u)du$$

$$=0+\int_{u=-a}^{x}\left(\frac{b}{a}u+b\right)du$$

$$=\left[\frac{b}{a}\frac{u^2}{2}+bu\right]_{-a}^{x}$$

$$=\frac{b}{a}\frac{x^2}{2}+bx-\left(\frac{b}{\not a}\frac{a^{\not 2}}{2}-ab\right)$$

$$=\frac{b}{2a}x^2+bx-\frac{ab}{2}+ab$$

$$=\frac{b}{2a}x^2+bx+\frac{ab}{2}$$

$$\therefore \quad \boxed{F_X(x)=\frac{b}{2a}x^2+bx+\frac{ab}{2}; \qquad -a \le x \le 0}$$

(iii) $F_X(x)$ for the interval $0 \le x \le a$

$$F_X(x)=\int_{x=-\infty}^{-a} f_X(x)dx+\int_{x=-a}^{0} f_X(x)dx+\int_{u=0}^{x} f_X(u)du$$

$$=0+\int_{x=-a}^{0}\left(\frac{b}{a}x+b\right)dx+$$

x

$\underline{b}$

$u{=}0$

—

u

a

$+\ b$

du

$\int$

$$= \left[\frac{b}{a}\frac{x^2}{2} + bx\right]_{-a}^{0} + \left[-\frac{b}{a}\frac{u^2}{2} + bu\right]_{0}^{x}$$

$$= \left[0 - \left(\frac{b}{a}\frac{(-a)^2}{2} + b(-a)\right)\right] + \left[-\frac{b}{a}\frac{x^2}{2} + bx - 0\right]$$

$$= \frac{-ab}{2} + ab - \frac{b}{a}\frac{x^2}{2} + bx$$

$$= -\frac{b}{2a}x^2 + bx + \frac{ab}{2}$$

$$\therefore \quad \boxed{F_X(x) = -\frac{b}{2a}x^2 + bx + \frac{ab}{2}; \qquad 0 \le x \le a}$$

(iv) $F_X(x)$ for the interval $a \le x \le +\infty$

$$F_X(x) = \int_{x=-\infty}^{-a} f_X(x)dx + \int_{x=-a}^{0} f_X(x)dx + \int_{x=0}^{a} f_X(x)dx + \int_{u=a}^{\infty} f_X(u)du$$

$$= 0 + \int_{x=-a}^{x}\left(\frac{b}{a}x + b\right)dx + \int_{x=0}^{a}\left(-\frac{b}{a}x + b\right)dx + 0$$

$$= \left[\frac{b}{a}\frac{x^2}{2} + bx\right]_{-a}^{0} + \left[-\frac{b}{a}\frac{x^2}{2} + bx\right]_{0}^{a}$$

$$= \left[0 - \left(\frac{b}{a}\frac{a^2}{2} - ab\right)\right] + \left[-\frac{b}{a}\frac{a^2}{2} + ab - 0\right]$$

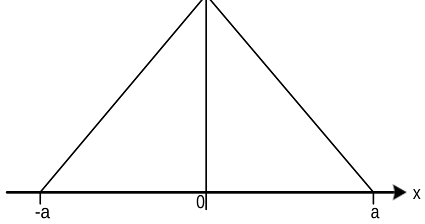
$$= -\frac{ab}{2} + ab - \frac{ab}{2} + ab$$

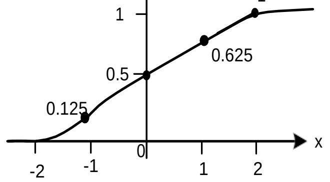
$$= -ab + 2ab$$

$$= ab$$

$$\therefore \quad \boxed{F_X(x) = ab; \qquad a \le x \le \infty}$$

$$F_X(x) = \begin{cases} 0; & -\infty \le x \le -a \\ \frac{b}{2a}x^2 + bx + \frac{ab}{2}; & -a \le x \le 0 \\ -\frac{b}{2a}x^2 + bx + \frac{ab}{2}; & 0 \le x \le a \\ ab; & a \le x \le \infty \end{cases}$$





Question. 7: A random variable has an experimental PDF: $f_X(x) = ae^{-b|x|}$ where a and b are constants. (i) Find the relation between a and b. (ii) Plot PDF and CDF.

Solution:

$$f_X(x) = \begin{cases} ae^{bx}; & -\infty \le x \le 0 \\ ae^{-bx}; & 0 \le x \le \infty \end{cases}$$

(i) We know that $\int_{x=-\infty}^{\infty} f_X(x)dx = 1$

$$\int_{x=-\infty}^{0} ae^{bx}dx + \int_{x=0}^{\infty} ae^{-bx}dx = 1$$

$$a\left[\frac{e^{bx}}{b}\right]_{-\infty}^{0} + a\left[\frac{e^{-bx}}{-b}\right]_{0}^{\infty} = 1$$

$$\frac{a}{b} - 0 + 0 + \frac{a}{b} = 1$$

$$\frac{2a}{b} = 1$$

$$\frac{a}{b} = \frac{1}{2}$$

$$\therefore \quad \boxed{\frac{a}{b} = \frac{1}{2}}$$

(ii) Find the expression for $F_X(x)$ in two intervals, i.e.,

(a) $-\infty \le x \le 0$; (b) $0 \le x \le +\infty$.

(a) $F_X(x)$ for the interval $-\infty \le x \le 0$

$$\begin{aligned} F_X(x) &= \int_{u=-\infty}^{x} f_X(u)du \\ &= \int_{u=-\infty}^{x} ae^{bu}du \\ &= a\left[\frac{e^{bu}}{b}\right]_{-\infty}^{x} \\ &= \frac{1}{2}\left[e^{bx} - 0\right] \\ &= \frac{1}{2}e^{bx} \end{aligned}$$

$$\therefore \quad \boxed{F_X(x) = \frac{1}{2}e^{bx}; \qquad -\infty \le x \le 0}$$

(b) $F_X(x)$ for the interval $0 \le x \le +\infty$.

$$\begin{aligned} F_X(x) &= \int_{x=-\infty}^{0} f_X(x)dx + \int_{x=0}^{x} f_X(u)du \\ &= \int_{x=-\infty}^{x} ae^{bx}dx + \int_{u=0}^{x} ae^{-bu}du \end{aligned}$$

$$= a\left[\frac{e^{bx}}{b}\right]_{-\infty}^{0} + a\left[\frac{e^{-bu}}{b}\right]_{0}^{x}$$

$$= \frac{a}{b}\left(1-0\right) - \frac{a}{b}\left(e^{-bx} - 1\right)$$

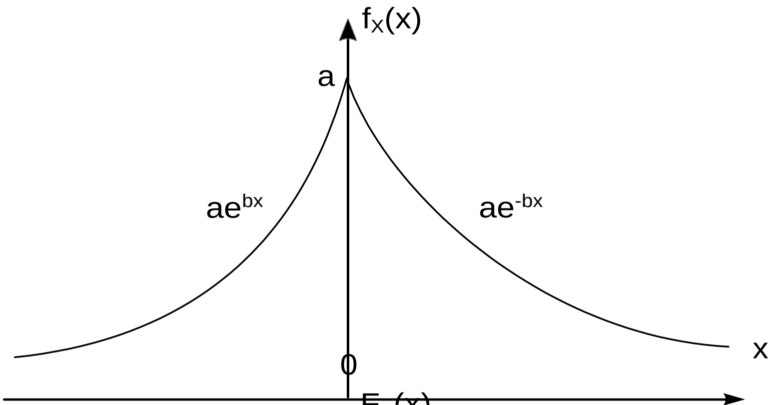
$$= \frac{a}{b} - \frac{a}{b}e^{-bx} + \frac{a}{b}$$

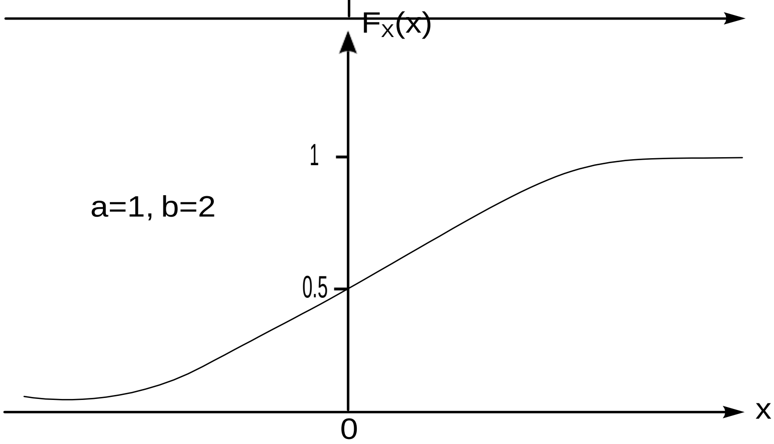
$$= \frac{2a}{b} - \frac{a}{b}e^{-bx}$$

$$= 1 - \frac{1}{2}e^{-bx}$$

$$\therefore \quad \boxed{F_X(x) = 1 - \frac{1}{2}e^{-bx}; \qquad 0 \le x \le +\infty}$$

$$F_X(x) = \begin{cases} \frac{1}{2}e^{bx}; & -\infty \le x \le 0 \\ 1 - \frac{1}{2}e^{-bx}; & 0 \le x \le +\infty \end{cases}$$





2.4 Statistical parameters of a Random variable

Expectation: It is process of averaging the random variable X.

Moment: Expected values of a function $g(x)$ of a random variable X is used for calculating the moment. Two types: (1) Moment about origin (2) Moment about mean.

(1) Moment about origin:

Let the function $g(x) = x^n;\ n = 0, 1, 2,$ The moment about origin can be written as

$$\boxed{m_n = E[X^n] = \overline{X^n} = \int_{x=-\infty}^{\infty} x^n f_X(x)dx}$$

- If $n = 0$ then $m_0 = \int_{x=-\infty}^{\infty} f_X(x)dx;$
  where m_0 is the *total area* of PDF curve.
- If $n = 1$ then $m_1 = E[X] = \overline{X} = \int_{x=-\infty}^{\infty} xf_X(x)dx = \overline{X};$
  where m_1 is called *mean value* of random variable X or (or the *expected* or the *average* or *D.C value* of X).
- If $n = 2$ then $m_2 = E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx;$
  where m_2 is called *mean square* value of random variable X, which is *total power* of random variable. i.e.,
  $m_2 = E[X^2] = Total\ Power = AC\ Power + DC\ Power$
- If $n = 3$ then $m_3 = E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx;$
  where m_3 is a 3^{rd} *moment about origion*.

(2) Moment about mean or Central moment:

Let the function $g(x) = (X - \overline{X})^n;\ n = 0, 1, 2, ...$ The *central moment* of a random variable X can be written as

$$\boxed{\mu_n = E[(x - \overline{X})^n] = \int_{x=-\infty}^{\infty} \left(x - \overline{X}\right)^n f_X(x)dx}$$

- If $n = 0$ then $\mu_0 = \int_{x=-\infty}^{\infty} f_X(x)dx;$
  where μ_0 is the *total area* of PDF curve.
- If $n = 1$ then

$$\mu_1 = E[(x - \overline{X})] = \int_{x=-\infty}^{\infty} (x - \overline{X})f_X(x)\,dx$$

$$= \int_{x=-\infty}^{\infty} xf_X(x)dx - \overline{X}\underbrace{\int_{x=-\infty}^{\infty} f_X(x)dx}_{=1}$$

$$= \overline{X} - \overline{X} = 0$$

$$\therefore \quad \boxed{\mu_1 = 0}$$

- If $n = 2$ then

$$\mu_2 = E[(x-\overline{X})^2] = \int_{x=-\infty}^{\infty} (x-\overline{X})^2 f_X(x)\,dx$$

$$= \sigma_X^2 = E\left[X^2 + \overline{X}^2 - 2X\overline{X}\right]$$

$$= E[X^2] + E[\overline{X}^2] - 2\overline{X}E[X]$$

$$= \overline{X^2} + \overline{X}^2 - 2\overline{X}^2$$

$$= \overline{X^2} - \overline{X}^2$$

$$= m_2 - m_1^2$$

$$\therefore \quad \boxed{\mu_2 = \sigma_X^2 = m_2 - m_1^2 = \text{mean square value} - \text{square of mean}}$$

The second central moment is called *variance*, and denoted by σ_X^2. Which is equal to AC power of random variable X.

In many practical problems, the measure of expected value $E(X)$ of a random variable 'X' does not completely describer or characterize the probability distribution. So, it is necessary to find spread or dispersion of the function about mean value. The quantity used to measure the width or spread or dispersion is called variance.

  - The $+ve$ square root of variance is called *standard deviation* σ_X of X. It is measure of the spread in the function $f_X(x)$ about the mean. $\sigma_X = \sqrt{m_2 - m_1^2}$
  - If $m_1 = 0$ then $\sigma_X = \sqrt{m_2} = \sqrt{\text{mean square}}$, It is called *root mean square value* or *r.m.s value or AC component* of random variable X

- If $n = 3$ then

$$\mu_3 = E[(x-\overline{X})^3] = \int_{x=-\infty}^{\infty} (x-\overline{X})^3 f_X(x)\,dx$$

$$= E\left[X^3 - 3X^2\overline{X} + 3X\overline{X}^2 - \overline{X}^3\right] \qquad \because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$= E[X^3] - 3\overline{X}E[X^2] + 3\overline{X}^2 E[X] - \overline{X}^3$$

$$= E[X^3] - 3\overline{X}(\sigma_X^2 + \overline{X}^2) + 3\overline{X}^3 - \overline{X}^3 \qquad \because m_2 = \sigma_X^2 + \overline{X}^2$$

$$= E[X^3] - 3\overline{X}\sigma_X^2 - 3\overline{X}^3 + 3\overline{X}^3 - \overline{X}^3$$

$$= \overline{X^3} - 3\overline{X}\sigma_X^2 - \overline{X}^3$$

$$= m_3 - 3m_1\sigma_X^2 - m_1^3$$

$$\therefore \quad \boxed{\mu_3 = \overline{X^3} - 3\overline{X}\sigma_X^2 - \overline{X}^3 = m_3 - 3m_1\sigma_X^2 - m_1^3}$$

The third central moment is called *skew* of PDF and it is measures asymmetry of $f_X(x)$ about mean. If a density function is symmetric about $x = \overline{X}$ then its skew is zero. The normalized third central moment, $\alpha_3 = \frac{\mu_3}{\sigma_X^3}$ is known as *skewness* of PDF or coefficient of skewness.

Summary:

Table 2.1 Summary of statistical parameters of a random variable X

S.No	Parameter	Mathematical Equation
1.	Mean Value	$m_1 = E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x) dx.$
2.	Mean square value	$m_2 = E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x) dx.$
3.	3^{rd} moment about origin	$m_3 = E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x) dx.$
4.	Variance	$\mu_2 = \sigma_X^2 = m_2 - m_1^2$
5.	Standard deviation	$\sigma_X = \sqrt{m_2 - m_1^2} = \sqrt{\mu_2}$
	r.m.s value of random variable X	If $m_1 = 0$, $\sigma_X = \sqrt{m_2}$
	standard deviation	If $m_1 \neq 0$, $\sigma_X = \sqrt{\mu_2}$
6.	Skew	$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$
	PDF symmetry about $\overline{X}$	If $\mu_3 = 0$
	PDF anti-symmetry about $\overline{X}$	If $\mu_3 \neq 0$
7.	Skewness or coefficient	$\alpha_3 = \frac{\mu_3}{\sigma_X^3}$

Question. 8: Find all statistical parameters of continuous random variable X, whose PDF is given by

$$f_X(x) = \begin{cases} \frac{2}{9}(x-1); & 1 \leq x \leq 4 \\ 0; & \text{else where} \end{cases}$$

Solution:

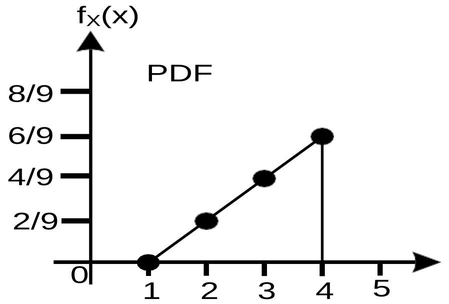
1. Mean vale $m_1 = E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x) dx.$

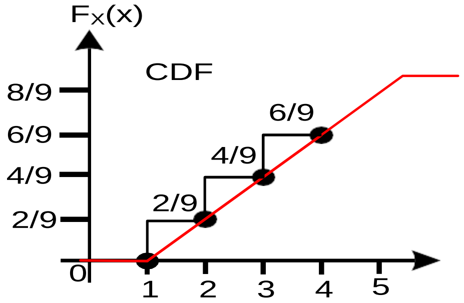
$$E[X] = \int_{x=-\infty}^{\infty} x \frac{2}{9}(x-1) dx = \frac{2}{9}\int_{x=1}^{4} (x^2 - x) dx$$

$$= \frac{2}{9}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^4$$

$$= \frac{2}{9}\left[\frac{4^3}{3} - \frac{4^2}{2} - \frac{1}{3} - \frac{1}{2}\right]$$

$$= \frac{2}{9} \times \frac{81}{6} = 3$$





2. Mean square value $m_2 = E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx.$

$$m_2 = \int_{x=-\infty}^{\infty} x^2 \frac{2}{9}(x-1)dx = \frac{2}{9}\int_{x=1}^{4}(x^3 - x^2)dx$$
$$= \frac{2}{9}\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^4$$
$$= \frac{2}{9}\left[\frac{4^4}{4} - \frac{4^3}{3} - \frac{1}{4} - \frac{1}{3}\right]$$
$$= \frac{2}{9}\left[\frac{255}{4} - \frac{63}{3}\right] = 9.5$$

3. 3^{rd} moment about origin $m_3 = E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx.$

$$m_3 = \int_{x=-\infty}^{\infty} x^3 \frac{2}{9}(x-1)dx = \frac{2}{9}\int_{x=1}^{4}(x^4 - x^3)dx$$
$$= \frac{2}{9}\left[\frac{x^5}{5} - \frac{x^4}{4}\right]_1^4$$
$$= \frac{2}{9}\left[\frac{4^5}{5} - \frac{4^4}{4} - \frac{1}{5} - \frac{1}{4}\right]$$
$$= \frac{2}{9}\left[\frac{1023}{5} - \frac{255}{4}\right] = 31.3$$

4. Variance $\mu_2 = \sigma_X^2 = m_2 - m_1^2 = 9.5^2 - 3^2 = 9.5 - 9 = 0.5$

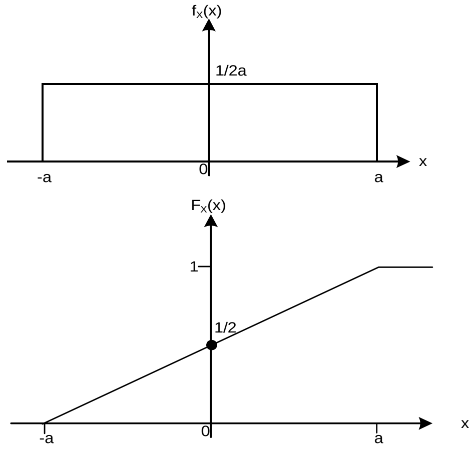
5. Standard deviation $\sigma_X = \sqrt{m_2 - m_1^2} = \sqrt{\mu_2} = \sqrt{0.5} = 0.7071$

6. Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 31.3 - 3(3)(0.5) - 3^3 = -0.2$

Question. 9: Find all statistical parameters of continuous random variable X, whose PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{2a}; & -a \leq x \leq a \\ 0; & \text{else where} \end{cases}$$

Solution:



1. Mean vale $m_1 = E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x) dx.$

$$E[X] = \int_{x=-\infty}^{\infty} x \frac{1}{2a} dx = \frac{1}{2a} \int_{x=-a}^{a} x dx = \frac{1}{2a} \left[\frac{x^2}{2}\right]_{-a}^{a} = \frac{1}{2a} \left[\frac{a^2}{2} - \frac{(-a)^2}{2}\right] = 0$$

2. Mean square value $m_2 = E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x) dx.$

$$m_2 = \int_{x=-\infty}^{\infty} x^2 \frac{1}{2a} dx = \frac{1}{2a} \int_{x=-a}^{a} x^2 dx$$

$$= \frac{1}{2a} \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2a} \left[\frac{a^3}{3} - \frac{(-a)^3}{3}\right] = \frac{1}{2a} \frac{2a^3}{3} = \frac{a^2}{3}$$

3. 3^{rd} moment about origin $m_3 = E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x) dx$

$$m_3 = \int_{x=-\infty}^{\infty} x^3 \frac{1}{2a} dx = \frac{1}{2a} \int_{x=-a}^{a} x^3 dx$$

$$= \frac{1}{2a} \left[\frac{x^4}{4}\right]_{-a}^{a} = \frac{1}{2a} \left[\frac{a^4}{4} - \frac{(-a)^4}{4}\right] = 0$$

4. Variance $\mu_2 = \sigma_X^2 = m_2 - m_1^2 = \frac{a^2}{3} - 0 = \frac{a^2}{3}$

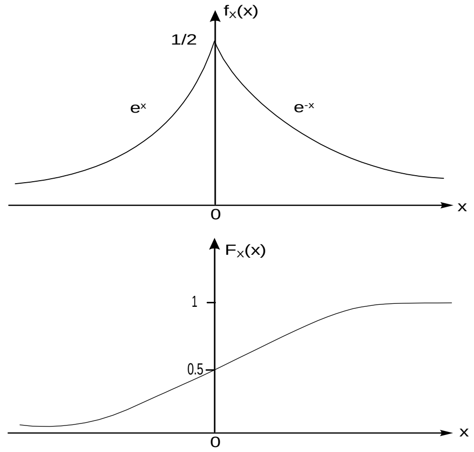
5. Standard deviation $\sigma_X = \sqrt{m_2 - m_1^2} = \sqrt{\mu_2} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}}$

6. Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 0 - 3(0)(\frac{a^2}{3}) - 0 = 0$

Conclusion: Skew is zero. So, the PDF function is symmetry.

Question. 10: Find all statistical parameters of continuous random variable X, whose PDF is given by $f_X(x) = \frac{1}{2}e^{|x|}; \quad -\infty \le x \le \infty$; where a and b are constants.

Solution: Given $f_X(x) = \frac{1}{2}e^{|x|}; \ -\infty \le x \le \infty;$



$$f_X(x) = \begin{cases} \frac{1}{2}e^{x}; & -\infty \le x \le 0 \\ \frac{1}{2}e^{-x}; & 0 \le x \le \infty \end{cases}$$

1. Mean vale $m_1 = E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x)dx.$

$$E[X] = \int_{x=-\infty}^{0} x f_X(x)dx + \int_{x=0}^{\infty} x f_X(x)dx$$

$$= \int_{x=-\infty}^{0} x\frac{1}{2}e^{x}dx + \int_{x=0}^{\infty} x\frac{1}{2}e^{-x}dx$$

$$= \frac{1}{2}\Big[e^{x}(x-1)\Big]_{-\infty}^{0} + \frac{1}{2}\Big[e^{-x}(-x-1)\Big]_{0}^{\infty}$$

$$= \frac{1}{2}\Big[e^{0}(0-1) - e^{-\infty}(-\infty-1)\Big] + \frac{1}{2}\Big[e^{-\infty}(-\infty-1) - e^{-0}(-0-1)\Big]$$

$$= \frac{1}{2}(-1-0) - \frac{1}{2}(0-1)$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

2. Mean square value $m_2 = E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx.$

$$m_2 = \int_{x=-\infty}^{0} x^2 f_X(x)dx + \int_{x=0}^{\infty} x^2 f_X(x)dx$$

$$= \int_{x=-\infty}^{0} x^2.\frac{1}{2}e^{x}dx + \int_{x=0}^{\infty} x^2.\frac{1}{2}e^{-x}dx$$

$$= \frac{1}{2}\Big[(x^2-2x+2)e^{x}\Big]_{-\infty}^{0} + \frac{1}{2}\Big[(-x^2-2x+2)e^{-x}\Big]_{0}^{\infty}$$

$$= \frac{1}{2}\Big[x^2e^x - 2xe^x - 2e^x\Big]_{-\infty}^{0} + \frac{1}{2}\Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_{0}^{\infty}$$

$$= \frac{1}{2}\Big[(0 - 0 + 2) - (0 - 0 + 0)\Big] + \frac{1}{2}\Big[(-0 - 0 - 0) - (0 - 0 - 2)\Big]$$

$$= \frac{1}{2}\Big[2\Big] + \frac{1}{2}\Big[(+2)\Big] = 1 + 1 = 2$$

3. 3^{rd} moment about origin $m_3 = E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx$

$$m_3 = \int_{x=-\infty}^{0} x^3.\frac{1}{2}e^x dx + \int_{x=0}^{+\infty} x^3.\frac{1}{2}e^{-x}dx$$

$$= \frac{1}{2}\Big[e^x(x^3 - 3x^2 + 6x - 6)\Big]_{-\infty}^{0} + \frac{1}{2}\Big[e^{-x}(-x^3 - 3x^2 - 6x - 6)\Big]_{0}^{\infty}$$

$$= \frac{1}{2}\Big[e^0(0 - 0 + 0 - 6)\Big] + \frac{1}{2}\Big[-(e^0(0 - 0 - 0 - 6))\Big]$$

$$= -\frac{6}{2} + \frac{6}{2} = 0$$

4. Variance $\mu_2 = \sigma_X^2 = m_2 - m_1^2 = 2 - 0 = 2$

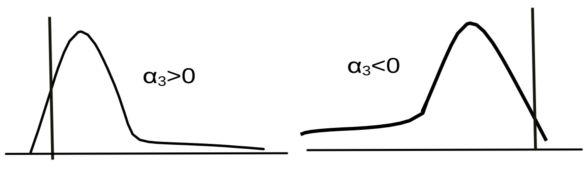
5. Standard deviation $\sigma_X = \sqrt{m_2 - m_1^2} = \sqrt{\mu_2} = \sqrt{2} = 1.414$

6. Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 0 - 0 - 0 = 0$

7. Skewness $\alpha_3 = \dfrac{\mu_3}{\sigma_X^3} = 0$ So, it is symmetric PDF function about mean value.

2.4.0.1 Skewness

Skewness of a probability distribution is a measure of asymmetry (or lack of symmetry). Recall that the probability distribution of random variable X is said to be symmetric about point $\overline{X}$. Often a distribution system is not symmetric about any value but instead has one of its tails longer than other.



- If the longer tail occurs to the right, the distribution is said to be skewed to the right.

- If the longer tail occurs to the left, the distribution is said to be skewed to the left.

- Measures describing this assymetry are called coefficients of skewness (or) briefly skewness. i.e, $\alpha_3 = \frac{E\left[(x-\mu)^3\right]}{\sigma^3} = \frac{E\left[(x-\overline{X})^3\right]}{\sigma^3} = \frac{\mu_3}{\sigma^3}$

- The measures α_3 will be positive then distribution is skewed to the right, α_3 will be negative then distribution is skewed to the left, and α_3 will be zero then ther PDF is a symmetric.

2.4.1 Properties of Expectation

Let 'X' be the random variable with PDF $f_X(x)$.
Expectation is defined as $E[X] = \overline{X} = m_1 = \int_{-\infty}^{\infty} x f_X(x)dx$

1. If random variable 'X' is constant then Expectation is also constant.

   Proof. $E[X] = \overline{X} = m_1 = \int_{-\infty}^{\infty} x f_X(x)dx$

   Let $X = \overline{K}$ = Constant

   $E[K] = X = m_1 = \int_{-\infty}^{\infty} K.f_X(x)dx = K.\underbrace{\int_{-\infty}^{\infty} f_X(x)}_{=1}$ □

   $\therefore E[K] = K$

2. Expectation of KX is $KE[X]$

   Proof. $E[KX] = \int_{-\infty}^{\infty} Kx.f_X(x)dx = K.\int_{-\infty}^{\infty} x f_X(x)dx = KE[X]$ □

3. $E[aX + b] = aE[X] + b$; where a an b are constants.

   Proof.

   $$E[aX + b] = \int_{-\infty}^{\infty} (ax + b)f_X(x)dx$$

   $$= a\int_{-\infty}^{\infty} x.f_X(x)dx + b\underbrace{\int_{-\infty}^{\infty} f_X(x)}_{=1}$$

   $$= aE[X] + b$$

   □

4. If x and y are two random variables with joint probability density function $f_{XY}(x, y)$ then $E[X + Y] = E[X] + E[Y]$

   Proof. Let X and Y be the two random variables with joint PDF $f_{XY}(x, y)$

   $$E[X + Y] = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} (x + y)f_{XY}(x, y)dxdy$$

   $$= \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} x f_{XY}(x, y)dxdy + \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} y f_{XY}(x, y)dxdy$$

   $$= \int_{x=-\infty}^{x=\infty} xdx \int_{y=-\infty}^{y=\infty} f_{XY}(x, y)dy + \int_{y=-\infty}^{y=\infty} ydy \int_{x=-\infty}^{x=\infty} f_{XY}(x, y)dx$$

$$= \int_{x=-\infty}^{x=\infty} x dx f_x(x) + \int_{y=-\infty}^{y=\infty} y dy f_Y(y)$$
$$= E[X] + E[Y]$$

□

5. If X and Y are two independent random variables with PDF $f_{XY}(x, y)$ then $E[XY] = E[X]E[Y]$

   Proof. We know that, if two random variable are independent then $f_{XY}(x, y) = f_X(x)f_Y(y)$ or $P(AB) = P(A)P(B)$

$$E[XY] = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} xyf_{XY}(x, y)dxdy$$
$$= \int_{x=-\infty}^{x=\infty} xf_X(x)dx \int_{y=-\infty}^{y=\infty} f_Y(y)dy$$
$$= E[X]E[Y]$$

□

6. If X and Y are two independent random variable such that $Y \leq X$ then $E[Y] \leq E[X]$

   Proof. $Y \leq X$
   $Y - X \leq 0$
   $E[Y - X] \leq E[0]$
   $E[y] - E[X] \leq 0$
   $E[Y] \leq E[X]$ Hence Proved. □

Question. 10: The PDF of continuous random variable is given by

$$f_X(x) = \begin{cases} \frac{2}{9}(x-1); & 1 \leq x \leq 4 \\ 0; & else\ where \end{cases}$$

Find $E[X]$, $E[X^2]$, $E[3X]$, $E[3X + 5]$, $E[(X - 1)^2]$
Solution:

1. $E[X] = \int_1^4 x.f_X(x)dx$

$$= \frac{2}{9}\int_1^4 x(x-1)dx$$
$$= \frac{2}{9}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^4$$

$$= \frac{2}{9}\Big[\Big(\frac{4^3}{3} - \frac{4^2}{2}\Big) - \Big(\frac{1}{3} - \frac{1}{2}\Big)\Big]$$

$$= \frac{2}{9}\Big[13.9\Big] = 3$$

2. $E[X^2] = \int_1^4 x^2.f_X(x)dx$

$$= \frac{2}{9}\int_1^4 x^2(x-1)dx$$

$$= \frac{2}{9}\Big[\frac{x^4}{4} - \frac{x^3}{3}\Big]_1^4$$

$$= \frac{2}{9}\Big[\Big(\frac{4^4}{4} - \frac{4^3}{3}\Big) - \Big(\frac{1}{4} - \frac{1}{3}\Big)\Big]$$

$$= \frac{2}{9}\Big[42.75\Big] = 9.5$$

3. $E[3X] = 3E[X] = 3 \times 3 = 9$

4. $E[3X + 5] = 3E[X] + 5 = 9 + 5 = 14$

5. $E[(X-1)^2] = E[X^2 - 2X + 1]$

$$= E[X^2] - 2E[X] + 1$$

$$= 9.5 - 2 \times 3 + 1$$

$$= 4.5$$

Question. 11: The PDF of continuous random variable is given by

$$f_X(x) = \begin{cases} 3x^2; & 0 \le x \le 1 \\ 0; & else\ where \end{cases}$$

Find (a) $E[X]$ (b) $E[X^2]$ (c) $E[3X-2]$

Solution:

(a) $E[X] = \int_{-\infty}^{\infty} x.f_X(x)dx = \int_0^1 x.3x^3dx = 3\Big[\frac{x^4}{4}\Big]_0^1 = \frac{3}{4}$

(b) $E[X^2] = \int_{-\infty}^{\infty} x^2.f_X(x)dx = \int_0^1 x^2.3x^3dx = 3\Big[\frac{x^5}{5}\Big]_0^1 = \frac{3}{5}$

(c) $E[3X-2] = 3E[X] - E[2] = 3\Big(\frac{3}{4}\Big) - 2 = \frac{1}{4}$

Question. 12: The continuous random variable 'X' is defined by

$$X = \begin{cases} -2 \text{ with a probability } 1/3; \\ 3 \text{ with a probability } 1/2; \\ 1 \text{ with a probability } 1/6; \end{cases}$$

Find (a) $E[X]$ (b) $E[X^2]$ (c) $E[2X + 5]$

Solution:

x_i	-2	3	1
$P(X_i) = f_X(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

$$(a).\ E[X] = \sum_x x f_X(x) = (-2)\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right) + 1\left(\frac{1}{6}\right) = 1$$

$$(b).\ E[X^2] = \sum_x x^2 f_X(x) = (-2)^2\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{2}\right) + 1^2\left(\frac{1}{6}\right) = 6$$

$$(c).\ E[2X + 5] = 2E[X] + 5 = 2(1) + 5 = 7$$

Question. 13: The PDF of continuous random variable is given by

$$f_X(x) = \begin{cases} 5e^{-5x}; & 0 \le x \le \infty \\ 0; & else\ where \end{cases}$$

Find (i) If PDF is valid. (ii) $E[X]$ (iii) $E[3X - 1]$ (iv) $E[(X - 1)^2]$

Solution: (i) The integration of PDF is equal to one, then it is valid.

$$Total\ Probability = 1$$

$$\int_{x=-\infty}^{\infty} f_X(x) = \int_0^{\infty} 5.e^{-5x} dx = 5\left[\frac{e^{-5}}{-5}\right]_0^{\infty} = -1\left[e^{-\infty} - e^{0}\right] = -1(-1)$$

$$= 1$$

So, the given PDF is valid.

$$(ii)\ E[X] = \int_0^{\infty} x.f_X(x)dx = 5\int_0^{\infty} x.e^{-5x}dx$$

$$= 5\left[\frac{xe^{-5x}}{-5} - \frac{1.e^{-5x}}{(-5)(-5)}\right]_0^{\infty} \qquad \because \boxed{\int uv = u\int v - \int du \int v}$$

$$= 5\left[0 - 0 - \left(0 - \frac{1}{25}\right)\right]$$

$$= \frac{1}{5}$$

$(ii)\ E[3X-1] = E[3X] - E[1] = 3E[X] - 1$

$$= 3\left(\frac{1}{5}\right) - 1 = \frac{2}{5}$$

$(iii)\ E[(X-1)^2] = E[X^2 - 2X + 1] = E[X^2] - 2E[X] + E[1]$

$$E[X^2] = \int_0^\infty x^2 . f_X(x) dx$$

$$= 5\int_0^\infty x^2 . e^{-5x} dx \qquad \because \boxed{\int uv = u\int v - \int du \int v}$$

$$= 5\left[x^2 \frac{e^{-5x}}{-5} - \int 2x \frac{e^{-5x}}{-5} dx\right]_0^\infty$$

$$= \left[x^2 e^{-5x}\right]_0^\infty + 2\left[\int xe^{-5x} dx\right]_0^\infty$$

$$= 0 + \frac{2}{25} = \frac{2}{25}$$

$\therefore\ E[(X-1)^2] = E[X^2] - 2E[X] + E[1]$

$$= \frac{2}{25} - 2\left(\frac{1}{5}\right) + 1$$

$$= \frac{2 - 10 + 25}{25}$$

$$= \frac{17}{25}$$

Question. 14: Let 'X' be the random variable defined by the density function

$$f_X(x) = \begin{cases} \frac{\pi}{16} cos(\frac{\pi x}{8}); & -4 \le x \le 4 \\ 0; & else\ where \end{cases}$$

Find $(i)\ E[3X]$ $\quad (ii)\ E[X^2]$

Solution:

$(i)\ E[3X] = 3E[X]$

$$E[X] = \int_{-\infty}^{\infty} x . f_X(x)\, dx$$

$$= \int_{-4}^{4} x . \frac{\pi}{16} cos(\frac{\pi x}{8}) dx$$

$$= \frac{\pi}{16}\int_{-4}^{4} x cos(\frac{\pi x}{8}) dx \qquad \because \boxed{\int uv = u\int v - \int du \int v}$$

$$= \frac{\pi}{16}\left[x \int cos(\frac{\pi x}{8}) - \int dx \int cos(\frac{\pi x}{8}) dx\right]_{-4}^{4}$$

$$= \frac{\pi}{16}\left[x \frac{sin(\frac{\pi x}{8})}{\frac{\pi}{8}} - \int 1. \frac{sin(\frac{\pi x}{8})}{\frac{\pi}{8}} dx\right]_{-4}^{4}$$

$$= \frac{\pi}{16}\left[\frac{x\sin(\frac{\pi x}{8})}{\frac{\pi}{8}} - \frac{-\cos(\frac{\pi x}{8})}{(\frac{\pi}{8})(\frac{\pi}{8})} \right]_{-4}^{4}$$

$$= \frac{\pi}{16} \times \frac{8}{\pi}\left[x\sin(\frac{\pi x}{8}) + \frac{8}{\pi}\cos(\frac{\pi x}{8}) \right]_{-4}^{4}$$

$$= \frac{1}{2}\left[x\sin(\frac{\pi x}{8}) \right]_{-4}^{4} + \frac{1}{2} \times \frac{8}{\pi}\left[\cos(\frac{\pi x}{8}) \right]_{-4}^{4}$$

$$= \frac{1}{2}\left[4\sin(\frac{4\pi}{8}) - 4\sin(\frac{-4\pi}{8}) \right] + \frac{4}{\pi}\left[\cos(\frac{4\pi}{8}) - \cos(\frac{4\pi}{8}) \right]$$

$$= \frac{1}{2} \times 8\sin\frac{\pi}{2} + \frac{4}{\pi}(0)$$

$$= \frac{1}{2}(0) + 0$$

$E[X] = 0$

$\therefore\ E[3X] = 3E[X] = 3(0) = 0$

$$(ii)\ E[X^2] = \int_{-\infty}^{\infty} x^2 . f_X(x)\, dx$$

$$= \int_{-4}^{4} x^2 . \frac{\pi}{16}\cos(\frac{\pi x}{8})dx \qquad \because \boxed{\int uv = u\int v - \int du \int v}$$

$$= \frac{\pi}{16}\int_{-4}^{4} x^2\cos(\frac{\pi x}{8})dx$$

$$= \frac{\pi}{16}\left[x^2 \frac{\sin(\frac{\pi x}{8})}{\frac{\pi}{8}} - \int 2x \frac{\sin(\frac{\pi x}{8})}{\frac{\pi}{8}} dx \right]_{-4}^{4}$$

$$= \frac{\pi}{16} \times \frac{8}{\pi}\left[x^2\sin(\frac{\pi x}{8}) - 2\int x\sin(\frac{\pi x}{8})dx \right]_{-4}^{4}$$

$$= \frac{1}{2}\left[x^2\sin(\frac{\pi x}{8}) - 2\left[x.\frac{-\cos(\frac{\pi x}{8})}{\frac{\pi}{8}} - \int 1.\frac{-\cos\frac{\pi x}{8}}{\frac{\pi}{8}}dx \right] \right]_{-4}^{4}$$

$$= \frac{1}{2}\left[x^2\sin(\frac{\pi x}{8}) - \frac{16}{\pi}\left[-x\cos(\frac{\pi x}{8}) + \frac{\sin(\frac{\pi x}{8})}{\frac{\pi}{8}} \right] \right]_{-4}^{4}$$

$$E[X^2] = \frac{1}{2}\left[\left[x^2\sin\frac{\pi x}{8}\right]_{-4}^{4} + \frac{16}{\pi}\left[x\cos\frac{\pi x}{8}\right]_{-4}^{4} - \frac{128}{\pi^2}\left[\sin\frac{\pi x}{8}\right]_{-4}^{4} \right]$$

$$= \frac{1}{2}(16 + 16) + \frac{16}{\pi}(0) - \frac{128}{\pi^2}(2)$$

$$= \frac{1}{2} \times 32 - \frac{1}{2} \times \frac{128}{\pi^2} \times 2$$

$$= 16 - \frac{128}{\pi^2}$$

2.4.2 Properties of Variance

Let 'X' be the random variable with PDF $f_X(x)$. Variance or Second central moment or ac power is defined as

$$Var(X) = \mu_2 = E[(x - \overline{X})^2] = \int_{x=-\infty}^{+\infty} \left(x - \overline{X}\right)^2 f_X(x)dx$$

1. $Var(X) = \sigma_X^2 = m_2 - m_1^2 = E[X^2] - (E[X])^2$

Proof.

$$\begin{aligned} Var(X) &= E[(X - \overline{X})^2] \\ &= E[X^2 + \overline{X}^2 - X\overline{X}] \\ &= E[X^2] + \overline{X}^2 - \overline{X}E[X] \\ &= E[X^2] - \overline{X}^2 - 2\overline{X}^2 \\ &= E[X^2] - \overline{X}^2 \end{aligned}$$

2. Variance of constant is zero.

Proof. Let $X = K$

$$\begin{aligned} Var(X) &= E[(X - \overline{X})^2] \\ Var(K) &= \sigma_K^2 = E[(K - K)^2] \\ &= E[(\overline{K} - \overline{K})^2] \qquad \because K = \overline{K} \\ &= 0 \end{aligned}$$

3. $\sigma_{KX}^2 = Var(KX) = K^2 Var(X)$ where 'K' is constant.

Proof.

$$\begin{aligned} \sigma_{KX}^2 &= Var(KX) = E[(KX - \overline{KX})^2] \\ &= E[(KX - K\overline{X})^2] \qquad \because \overline{KX} = E[KX] = K\overline{X} \\ &= E[(KX - K\overline{X})^2] \\ &= E[K^2(X - \overline{X}^2)] \\ Var(KX) &= K^2 E[X - X^2] \end{aligned}$$

4. $\sigma_{aX+b}^2 = Var(aX + b) = a^2 Var(X)$

Proof.

$$\begin{aligned} Var(aX + b) &= E\left[[(aX + b) - (\overline{aX + b})]^2\right] \\ &= E\left[[(aX + b) - (a\overline{X} + b)]^2\right] \qquad \because \overline{aX + b} = E[aX + b] = a\overline{X} + b \\ &= E[a^2(X - \overline{X})^2] \end{aligned}$$

$$= a^2 Var(X)$$

5. $\sigma^2_{X+Y} = Var(X+Y) = Var(X) + Var(Y)$; If X, Y r.v are independent

Proof.

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y) - (\overline{X+Y})]^2 \\
&= E\left[((X+Y) - (\bar{X} + \bar{Y}))^2\right] \qquad \because \overline{X+Y} = E[X+Y] = \bar{X} + \bar{Y} \\
&= E\left[((X - \bar{X}) + (Y - \bar{Y}))^2\right] \\
&= E[(X - \bar{X})^2] + E[(Y + \bar{Y})^2] + 2E[X - \bar{X}]\,E[Y - \bar{Y}] \\
&= E[(X - \bar{X})^2] + E[(Y + \bar{Y})^2] \qquad \because E[X - \bar{X}] = \bar{X} - \bar{X} = 0 \\
&= Var(X) + Var(Y)
\end{aligned}
$$

6. $\sigma^2_{X-Y} = Var(X-Y) = Var(X) + Var(Y)$; If X, Y r.v are independent

Proof.

$$
\begin{aligned}
Var(X-Y) &= Var(X + (-Y)) \\
&= Var(1.X) + Var(-Y) \\
&= 1^2 Var(X) + (-1)^2 Var(Y) \\
&= Var(X) + Var(Y)
\end{aligned}
$$

7. If X, Y are two independent random variable with joint PDF $f_{XY}(x, y)$ then $Var(XY) = E[X^2]E[Y^2] - \bar{X}^2\bar{Y}^2$

Proof.

$$
\begin{aligned}
Var(XY) &= E[(XY - \overline{XY})^2] \\
&= E[(XY - \bar{X}\bar{Y})^2] \qquad \because \overline{XY} = E[XY] = E[X]E[Y] = \bar{X}\bar{Y} \\
&= E[X^2Y^2 + \bar{X}^2\bar{Y}^2 - 2XY\bar{X}\bar{Y}] \\
&= E[X^2]E[Y^2] + \bar{X}^2\bar{Y}^2 - 2\bar{X}^2\bar{Y}^2 \\
&= E[X^2]E[Y^2] - \bar{X}^2\bar{Y}^2
\end{aligned}
$$

8. $\sigma^2_{X+Y} = Var(X+Y) = Var(X) + Var(Y)$

Proof.

$$
\begin{aligned}
Var(X+Y) &= E[(X+Y) - (\overline{X+Y})]^2 \\
&= E\left[((X+Y) - (\bar{X} + Y))^2\right] \\
&= E\left[((X - \bar{X}) + (Y - \bar{Y}))^2\right]
\end{aligned}
$$

$$= E\left[(X - \overline{X})^2 + (Y - \overline{Y})^2 + 2(X - \overline{X})(Y - \overline{Y})\right]$$

$$= E\left[(X - \overline{X})^2 + (Y - \overline{Y})^2 + 2(XY - X\overline{Y} - \overline{X}Y + \overline{X}\,\overline{Y})\right]$$

$$= E[(X - \overline{X})^2] + E[(Y + \overline{Y})^2] + 2\left(E[XY] - E[X\overline{Y}] - E[\overline{X}Y] + E[\overline{X}\,\overline{Y}]\right)$$

$$= Var(X) + Var(Y) + 2\left(E[XY] - \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} + \overline{X}\,\overline{Y}\right)$$

$$= Var(X) + Var(Y) + 2\left(E[XY] - \overline{X}\,\overline{Y}\right)$$

If two r.v X, Y are independent, then $E[XY] = E[X]E[Y] = \overline{X}\,\overline{Y}$

$$Var(X + Y) = Var(X) + Var(Y) + 2\left(\overline{X}\,\overline{Y} - \overline{X}\,\overline{Y}\right)$$

$$Var(X + Y) = Var(X) + Var(Y)$$

Similarly

$Var(X - Y) = Var(X) + Var(Y) - 2\left(E[XY] - \overline{X}\,\overline{Y}\right)$

$Var(X - Y) = Var(X) + Var(Y)$ for independence r.v case.

Question. 15: If a random variable'X' is such that $E[(X - 1)^2] = 10$ and $E[(X - 2)^2] = 6$ then find (a) $E[X]$ $(b) Var(X)$

Solution:

(a)

$$E[(X - 1)^2 = E[X^2 + 1 - 2X] = 10 \longrightarrow E[X^2] - 2E[X] + 1 = 10 \qquad (2.1)$$

$$E[(X - 2)^2 = E[X^2 + 4 - 4X] = 6 \longrightarrow E[X^2] - 4E[X] + 4 = 6 \qquad (2.2)$$

By adding eqn.2.1 and eqn.2.2 $\longrightarrow 2E[X] - 3 = 4;$ $E[X^2] = 16$

$$\therefore E[X] = \frac{7}{2}$$

$$(b)\ Var(X) = m_2 - m_1^2 = 16 - \left(\frac{7}{2}\right)^2 = 16 - \frac{49}{4} = \frac{64 - 49}{4} = \frac{15}{4}$$

2.5 Standard PDF and CDF for Continues Random Variable (or) Different types of PDF and CDF

2.5.1 Uniform PDF

A continuous random variable X is said to follow a uniform distribution $[a, b]$ if its PDF is $f_X(x) = K; \quad a \le x \le b$

$$f_X(x) = \begin{cases} \frac{1}{b-a}; & a \le x \le b \\ 0; & Else\ where \end{cases}$$

Proof. From figure.

$$f_X(x) = \begin{cases} c; & a \le x \le b \\ 0; & Else\ where \end{cases}$$

We know that, the area under the curve is unity, i.e., $\int_{x=-\infty}^{\infty} f_X(x)dx = 1.$

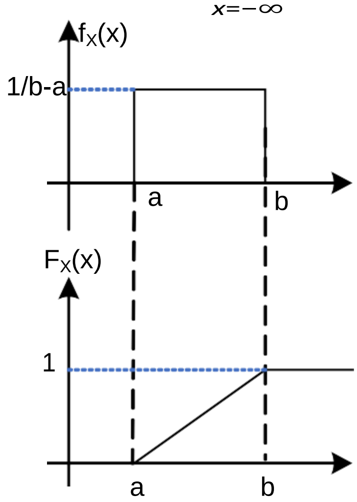
$$\int_{x=a}^{b} f_X(x)dx = 1$$

$$\int_{x=a}^{b} c\,dx = 1$$

$$c[x]_a^b = 1$$

$$c[b-a] = 1$$

$$c = \frac{1}{b-a}$$



□

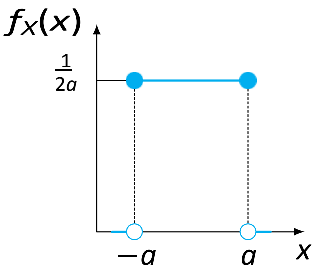
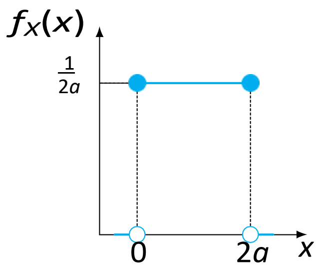
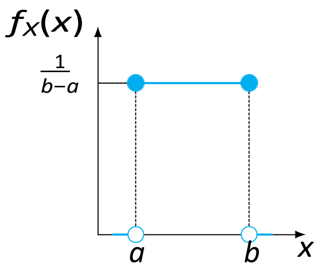
Application: It is used to represent the quantization noise in digital communication systems. Quantization is roundoff process in which actual sample value is rounded to nearest quantization level.

Quantization noise = Actual sample − rounded noise.

2.5.1.1 Statistical Parameters for Uniform PDF

Question. 16: Calculate all statistical parameters for uniform random variable, whose PDF is shown in Fig.

Solution:

 Fig. (a) Expectation: $E[X] = \overline{X}$	 Fig. (a) Expectation: $E[X] = \overline{X}$	 Fig. (a) Expectation: $E[X] = \overline{X}$
$m_1 = \int_{x=-a}^{a} x \cdot f_X(x)dx$ $= \int_{x=-a}^{a} \frac{1}{2a} \cdot xdx$ $= \frac{1}{2a}\left[\frac{x^2}{2}\right]_{-a}^{a}$ $= 0$ Mean square: $E[X^2]$	$m_1 = \int_{x=0}^{2a} x \cdot f_X(x)dx$ $= \int_{x=0}^{2a} \frac{1}{2a} \cdot xdx$ $= \frac{1}{2a}\left[\frac{x^2}{2}\right]_{0}^{2a}$ $= \frac{1}{2a}\frac{4a^2-0}{2}$ $= a$ Mean square: $E[X^2]$	$m_1 = \int_{x=a}^{b} x \cdot f_X(x)dx$ $= \int_{x=a}^{b} \frac{1}{b-a} \cdot xdx$ $= \frac{1}{2a}\left[\frac{x^2}{2}\right]_{a}^{b}$ $= \frac{1}{2a}\frac{b^2-a^2}{2}$ $= \frac{a+b}{2}$ Mean square: $E[X^2]$
$m_2 = \int_{x=-a}^{a} x^2 \cdot f_X(x)dx$ $= \int_{x=-a}^{a} \frac{1}{2a} \cdot x^2dx$ $= \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a}$ $= \frac{1}{6a}\left[a^3 + a^3\right]$ $= \frac{a^2}{3}$ Variance: $E[(X-\overline{X})^2]$	$m_2 = \int_{x=0}^{2a} x^2 \cdot f_X(x)dx$ $= \int_{x=0}^{2a} \frac{1}{2a} \cdot x^2dx$ $= \frac{1}{2a}\left[\frac{x^3}{3}\right]_{0}^{2a}$ $= \frac{1}{6a}\frac{8a^3}{3}$ $= \frac{4a^2}{3}$ Variance: $E[(X-\overline{X})^2]$	$m_2 = \int_{x=a}^{b} x^2 \cdot f_X(x)dx$ $= \int_{x=a}^{b} \frac{1}{b-a} \cdot x^2dx$ $= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_{a}^{b}$ $= \frac{1}{b-a}\frac{b^3-a^3}{3}$ $= \frac{b^3-a^3}{3(b-a)}$ Variance: $E[(X-\overline{X})^2]$
$\mu_2 = \sigma_X^2 = m_2 - m_1^2$ $= \frac{a^2}{3} - 0$ $= \frac{a^2}{3}$	$\mu_2 = \sigma_X^2 = m_2 - m_1^2$ $= \frac{4a^2}{3} - a^2$ $= \frac{a^2}{3}$	$\mu_2 = \sigma_X^2 = m_2 - m_1^2$ $= \frac{b^3-a^3}{3(b-a)} - (\frac{a+b}{2})^2$ $= \frac{(b-a)^2}{12}$

3rd momentum: $E[X^2]$	3rd momentum: $E[X^2]$	3rd momentum: $E[X^2]$
$m_3 = \int_{x=-a}^{a} x^3 \cdot f_X(x)dx$ $= \int_{x=-a}^{a} \frac{1}{2a} \cdot x^3 dx$ $= \frac{1}{2a}\left[\frac{x^4}{4}\right]_{-a}^{a}$ $= 0$	$m_3 = \int_{x=0}^{2a} x^3 \cdot f_X(x)dx$ $= \int_{x=0}^{2a} \frac{1}{2a} \cdot x^3 dx$ $= \frac{1}{2a}\left[\frac{x^4}{4}\right]_{0}^{2a}$ $= \frac{1}{2a}\left[\frac{16a^4-0}{4}\right]$ $= 2a^3$	$m_3 = \int_{x=a}^{b} x^3 \cdot f_X(x)dx$ $= \int_{x=a}^{b} \frac{1}{b-a} \cdot x^3 dx$ $= \frac{1}{b-a}\left[\frac{x^4}{4}\right]_{a}^{b}$ $= \frac{1}{b-a}\left[\frac{b^4-a^4}{4}\right]$ $= \frac{b^4-a^4}{4(b-a)}$
Standard deviation: σ_X	Standard deviation: σ_X	Standard deviation: σ_X
$\sqrt{\mu_2} = \sqrt{m_2 - m_1^2}$ $= \sqrt{\frac{a^2}{3}}$ $\sigma_X = \frac{a}{\sqrt{3}}$	$\sqrt{\mu_2} = \sqrt{m_2 - m_1^2}$ $= \sqrt{\frac{a^2}{3}}$ $\sigma_X = \frac{a}{\sqrt{3}}$	$\sqrt{\mu_2} = \sqrt{m_2 - m_1^2}$ $= \sqrt{\frac{(b-a)^2}{12}}$ $\sigma_X = \frac{b-a}{\sqrt{12}}$
Skew: μ_3	Skew: μ_3	Skew: μ_3
$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 0 - 0 - 0$ $= 0$	$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 2a^3 - 3a\frac{a^2}{3} - a^3$ $= 0$	$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 0$

NOTE: The mean value locates for continuous r.v, the center of gravity of the area under the PDF curve.

Question. 17: Let 'X' is uniform random variable, which represents Quantization Noise Power (QNP) and defined as

$$f_X(x) = \begin{cases} \frac{1}{20}; & 0 \le x \le 20 \\ 0; & \text{Else where} \end{cases}$$

1. Find average QNP?
2. What is the probability the QNP is greater than the average power?
3. What is the probability the QNP is ± 5 about the average power?

Solution:

1. Average QNP $E[X] = \int_{x=-\infty}^{\infty} x f_X(x)dx$

$$= \int_{x=0}^{20} \frac{1}{20} x dx$$

$$= \frac{1}{20}\left[\frac{x^2}{2}\right]_0^{20}$$

$$= \frac{1}{20}\left[\frac{20\times 20}{2}\right]$$

$$= 10$$

2. Probability the QNP is greater than the average power

$$P\{X \ge 10\} = \int_{x=10}^{20} f_X(x)dx$$

$$= \int_{x=10}^{20} \frac{1}{20} dx$$

$$= \frac{1}{20}\left[x\right]_{10}^{20}$$

$$= \frac{1}{20}\left[20 - 10\right]$$

$$= \frac{1}{2}$$

3. Probability that QNP ± 5 about the average power.

$$P\{5 \le X \le 15\} = \int_{x=5}^{15} f_X(x)dx$$

$$= \int_{x=5}^{15} \frac{1}{20} dx$$

$$= \frac{1}{20}\left[x\right]_{5}^{15}$$

$$= \frac{1}{20}\left[15 - 5\right]$$

$$= \frac{1}{2}$$

Question. 18: 'X' is a continuous random variable $X(\vartheta) = A\cos\vartheta$; which PDF is a uniform $(0,2\pi)$ random variable. Find the mean value of r.v?

Solution:

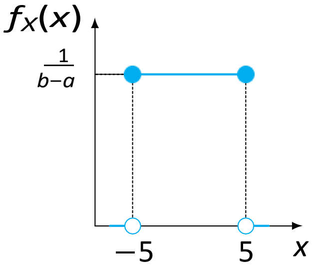
$$E[X(\vartheta)] = \overline{X(\vartheta)}$$

$$= \int_{x=-\infty}^{\infty} X(\vartheta) f_X(\vartheta) d\vartheta$$

$$= \frac{1}{2\pi}\int_{x=0}^{2\pi} A\cos\vartheta d\vartheta$$

$$= 0$$

Question. 19: 'X' is a continuous random variable $(-5, 5)$;



1. What is the PDF of 'X' $f_X(x)$?
2. What is the CDF of 'X' $F_X(x)$?
3. what is the $E[X]$, $E[X^5]$, $E[e^x]$?

Solution:

1. PDF function $f_X(x)$

$$f_X(x) = \frac{1}{b-a}$$

$$= \frac{1}{10}$$

$$f_X(x) = \frac{1}{10}; \quad -5 \le x \le 5$$

$$= 0; \quad \text{Else where}$$

2. CDF function $F_X(x)$

$$F_X(x) = \frac{x-a}{b-a}; \quad a \le x \le b$$

$$F_X(x) = \frac{x-(-5)}{5-(-5)} \quad -5 \le x \le 5$$

$$\therefore F_X(x) = \frac{x+5}{10} \quad -5 \le x \le 5$$

$$= 0; \quad \text{Else where}$$

3. $E[X] = \int_{x=-\infty}^{\infty} x f_X(x)dx = \int_{x=-5}^{5} x \frac{1}{10} dx = 0$ (odd function)

$E[X^5] = \int_{x=-\infty}^{\infty} x^5 f_X(x)dx = \int_{x=-5}^{5} x^5 \frac{1}{10} dx = 0$ (odd function)

$E[e^x] = \int_{x=-\infty}^{\infty} e^x f_X(x)dx = \int_{x=-5}^{5} e^x \frac{1}{10} dx = \frac{1}{10} \left[e^x \right]_{-5}^{5} = 14.84$

Question. 19: 'X' is a uniform random variable with expected value $E[X] = 7$ and variance $Var[X] = 3$. What is the PDF of 'X'?

Solution:

$$E[X] = \frac{a+b}{2} = 7$$

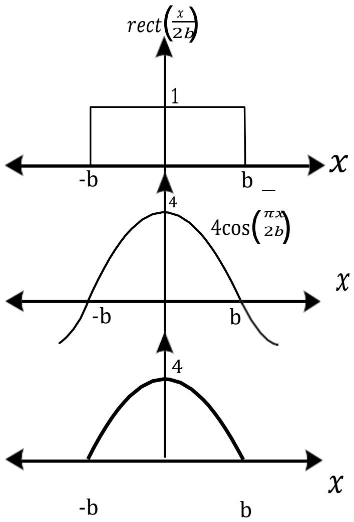
$$Var[X] = \sigma_X^2 = \frac{(b-a)^2}{12} = 3 \Rightarrow (b = a) = 6$$

From the above equations, $a = 4$; $b = 10$

$$\therefore f_X(x) = \begin{cases} \frac{1}{6}; & 4 \le x \le 10 \\ 0; & \text{Else where} \end{cases}$$

Question. 19: Given the function $g_X(x) = 4cos(\frac{\pi x}{2b})rect(\frac{x}{2b})$. Find the range values of 'b'which $g_X(x)$ is valid?

Solution:



We know that $\int_{x=-\infty}^{\infty} g_X(x)dx = 1$

$$\Rightarrow \int_{x=-b}^{b} 4cos(\frac{\pi x}{2b})rect(\frac{x}{2b})dx = 1$$

$$\Rightarrow \int_{x=-b}^{b} 4cos(\frac{\pi x}{2b})dx = 1$$

$$\Rightarrow 4 \times \frac{2b}{\pi}\left[sin\frac{\pi x}{2b}\right]_{-b}^{b} = 1$$

$$\Rightarrow b = \frac{\pi}{16}$$

2.5.2 Exponential random Variable

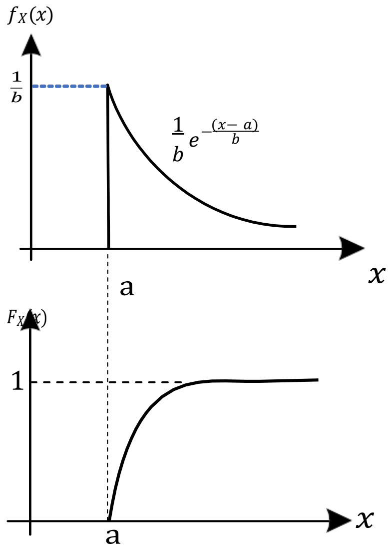
An exponential distribution function (PDF) can be defined for a continuous random variable X is

$$f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-(x-a)}{b}}; & x \geq a \\ 0; & x < a \end{cases}$$

The cummulative distribution function (CDF) is

$$\begin{aligned} F_X(x) &= \int_{u=-\infty}^{x} f_X(u)du \\ &= \int_{u=a}^{x} \frac{1}{b} e^{\frac{-(u-a)}{b}} du \\ &= \frac{1}{b} \left[\frac{e^{\frac{-(u-a)}{b}}}{-\frac{1}{b}} \right]_a^x \\ &= 1 - e^{\frac{-(x-a)}{b}} \end{aligned}$$

$$\therefore F_X(x) = \begin{cases} 1 - e^{\frac{-(x-a)}{b}}; & x \geq a \\ 0; & x < a \end{cases}$$



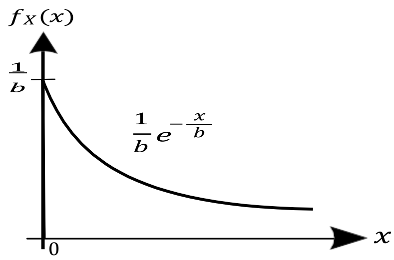
Applications:

- The exponential density function is useful in describing rain drop sizes when a large number of strome measurements are made.
- Describes the fluctuation in signal strength received by RADAR from certain types of air-cradft
- In communication systems, if occurance of events over non-overlapping intervals are independent, such as arrivel times of telephone calls or bus arrival times at a bus-stop, then the waiting time distribution of these events can be shown to be exponentional.

2.5.2.1 Statistical Parameters for Exponential PDF

Question. 20: Calculate all the statistical averages or parameters for exponetional PDF as shown in Fig.

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x)}{b}}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$



Solution:

1. Mean Value $E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x)dx$

$$m_1 = \int_{x=0}^{\infty} x \frac{1}{b} e^{-x/b} dx$$

$$= \frac{1}{b}\left[x\frac{e^{-x/b}}{-1/b} - \int \frac{e^{-x/b}}{-1/b} \right]_0^\infty$$

$$= \frac{1}{b}\left[x\frac{e^{-x/b}}{-1/b} - \int 1.\frac{dx}{1/b} \right]_0^\infty$$

$$= \frac{1}{b}\left[0 - 0 - (0 - b^2)\right]$$

$$= b$$

2. Mean Square Value $E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx$

$$m_2 = \int_{x=0}^{\infty} x^2 \frac{1}{b} e^{-x/b} dx$$

$$= \frac{1}{b}\left[x^2\frac{e^{-x/b}}{-1/b} - \int 2x\frac{e^{-x/b}}{-1/b}dx \right]_0^\infty$$

$$= \frac{1}{b}\left[-bx^2 e^{-x/b}\right]_0^\infty{}^{=0} + \left[\frac{2b}{b}\int_0^\infty xe^{-x/b}dx\right]$$

$$= 2b^2 \quad \because \text{ from mean value calculation } \int_0^\infty xe^{-x/b}dx = b^2$$

3. 3rd momentum $E[X^3] = \overline{X^3} = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx$

$$m_3 = \int_{x=0}^{\infty} x^3 \frac{1}{b} e^{-x/b} dx$$

$$= \frac{1}{b}\left[x^3\frac{e^{-x/b}}{-1/b} - \int 3x^2\frac{e^{-x/b}}{-1/b}dx \right]_0^\infty$$

$$= \frac{1}{b}\left[-bx^3 e^{-x/b}\right]_0^\infty{}^{=0} + \left[\frac{3b}{b}\int_0^\infty x^2e^{-x/b}dx\right]$$

$$= 3\left[\int_0^\infty x^2e^{-x/b}dx\right]$$

$$= 3\left[x^2\frac{e^{-x/b}}{-1/b}\Big|_0^\infty - \int_0^\infty \frac{2xe^{-x/b}}{\frac{-1}{b}}dx\right]$$

$$= 3\left[x^2\frac{e^{-x/b}}{-1/b}\Big|_0^\infty{}^{=0} + \int_0^\infty 2bxe^{-x/b}dx\right]$$

$$= 6b(b^2) \quad \because \text{ from mean value calculation } \int_0^\infty xe^{-x/b}dx = b^2$$

$$= 6b^3$$

4. Variance $\mu_2 = \sigma_X^2 = m_2 - m_1^2 = 2b^2 - b^2 = b^2$

5. Standard Deviation $\sigma_X = \sqrt{\mu_2} = \sqrt{b^2} = b$

6. Skew $\mu_3 = m_3 - 3m_1\mu_2 - m^3 = 6b^2 - 3bb^2 - b^3 = 2b^3$

Question. 21: Calculate all the statistical parameters for given exponetional PDF.

$$f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-(x-a)}{b}}; & x \geq a \\ 0; & x < a \end{cases}$$

Solution:

1. Mean Value $E[X] = \overline{X} = \int_{x=-\infty}^{\infty} x f_X(x)dx$

$$m_1 = \int_{x=a}^{\infty} x \cdot \frac{1}{b} e^{\frac{-(x-a)}{b}} dx$$

$$= \frac{1}{b}\left[\left(x \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b}\right) - \int 1 \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b} dx\right]_a^{\infty}$$

$$= \frac{1}{b}\left[(-b)\left(xe^{-\infty} - ae^0\right) + b\left[\frac{e^{\frac{-(x-a)}{b}}}{-1/b}\right]_a^{\infty}\right]$$

$$= \frac{1}{b}\left[(-b)\left[0 - a\right] - b^2\left[e^{-\infty} - e^0\right]\right]$$

$$= \frac{1}{b}\left[ab - b^2(0 - 1)\right]$$

$$= \frac{1}{b}\left[ab + b^2\right]$$

$E[X] = a + b$

2. Mean Square Value $E[X^2] = \overline{X^2} = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx$

$$m_2 = \int_{x=a}^{\infty} x^2 \cdot \frac{1}{b} e^{\frac{-(x-a)}{b}} dx$$

$$= \frac{1}{b}\left[\left(x^2 \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b}\right) - \int 2x \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b} dx\right]_a^{\infty}$$

$$= \frac{1}{b}\left[(-b)\left(x^2 e^{-\infty} - a^2 e^0\right) + 2b\left[\int_{x=a}^{\infty} x e^{\frac{-(x-a)}{b}} dx\right]\right]$$

$$= \frac{1}{b}\left[(-b)\left[0 - a^2\right] + 2b\left[b^2 + ab\right]\right] \qquad \because \frac{1}{b}\int_{x=a}^{\infty} x e^{\frac{-(x-a)}{b}} dx = a + b$$

$$= \frac{1}{b}\left[a^2b + 2b^3 + 2ab^2\right] \qquad \Rightarrow \int_{x=a}^{\infty} x e^{\frac{-(x-a)}{b}} dx = ab + b^2$$

$E[X^2] = a^2 + 2ab + 2b^2$

3. 3rd momentum $E[X^3] = \overline{X} = \int_{x=-\infty}^{\infty} x^3 f_X(x) dx$

$$m_3 = \int_{x=a}^{\infty} x^3 \cdot \frac{1}{b} e^{\frac{-(x-a)}{b}} dx$$

$$= \frac{1}{b}\left[x^3 \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b} - \int 3x \cdot \frac{e^{\frac{-(x-a)}{b}}}{-1/b} dx \right]_a^{\infty}$$

$$= \frac{1}{b}\left[(-b)\left(x^3 e^{-\infty} - a^3 e^0\right) + 3b \int_{x=a}^{\infty} x^2 e^{\frac{-(x-a)}{b}} dx\right]$$

$$= \frac{1}{b}\left[(-b)\left(0 - a^3\right) + 3b\left(a^2b + 2b^3 + 2ab^2\right)\right] \qquad \because \frac{1}{b}\int_{x=a}^{\infty} x^2 e^{\frac{-(x-a)}{b}} dx$$

$$= \frac{1}{b}\left[a^3b + 3a^2b^2 + 6b^4 + 6ab^3\right] \qquad = a^2 + 2b^2 + 2ab$$

$E[X^3] = a^3 + 3a^2b + 6b^3 + 6ab^2$

4. Variance: $\mu_2 = \sigma_X^2 = m2 - m^2{}_1$

$$= a^2 + 2ab + 2b^2 - (a + b)^2$$
$$= a^2 + 2ab + 2b^2 - a^2 - b^2 - 2ab$$
$$= b^2$$

5. Standatd deviation: $\sigma_X = \sqrt{\mu_2} = \sqrt{b^2} = b$

6. Skew: $\mu_3 = m_3 - 3m_1\mu_2 - m^2{}_1$

$$= a^3 + 3a^2b + 6b^3 + 6ab^2 - 3(a + b)b^2 - (a + b)^3$$
$$= a^3 + 3a^2b + 6b^3 + 6ab^2 - 6ab^2 - 4b^3 - a^3 - 3a^2b$$
$$= 2b^3$$

7. Skewness: $\frac{\mu_3}{\sigma_X^3} = \frac{2b^3}{b^3} = 2$

Question. 22: The power reflected from an aircraft of complicated shape that is received by a RADAR can be described by an exponentional random variable P , the density of 'P' is given by

$$f_P(p) = \begin{cases} \frac{1}{p_0} e^{\frac{-p}{p_0}}; & p \geq 0 \\ 0; & p < 0 \end{cases}$$

where p_0 is the average amount received power. What is probability that the received power is larger than the power received on the average?

Solution:

$$F_P(p) = \begin{cases} 1 - \frac{1}{p_0} e^{\frac{-p}{p_0}}; & p \geq 0 \\ 0; & p < 0 \end{cases}$$

$$\begin{aligned} F_P(-\infty \leq P \leq p_0) &= 1 - e^{\frac{-p_0}{p_0}} \\ &= 1 - e^{-1} \\ &= 0.632 \end{aligned}$$

Above the average power $= 1 - F_P(-\infty \leq P \leq p_0) = 0.368$

The received power is larger that its average value about 36.8% of the time.

Question. 23: The power reflected from an aircraft of completed shape that is received by a RADAR can be described by an exponentional random variable X, the PDF is described by

$$f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-x}{b}}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

- Find the average power (reflected power)?
- Find the probability that the received power is greater than the average power?

Solution:

1. Average power

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &= \int_{x=0}^{\infty} x \cdot \frac{1}{b} e^{\frac{x}{b}}\,dx \\ &= \frac{1}{b}\left[x \cdot \frac{e^{\frac{-x}{b}}}{-1/b} - \int 1 \cdot \frac{e^{\frac{-x}{b}}}{-1/b}\,dx \right]_0^{\infty} \\ &= \frac{1}{b}(-b)\left[xe^{-\infty} - 0e^{0} \right] + b\left[\frac{e^{\frac{-x}{b}}}{-\frac{1}{b}} \right]_0^{\infty} \end{aligned}$$

$$= \frac{1}{b}\left[(-b)\left[0-0\right] - b^2\left[e^{-\infty} - e^0\right]\right]$$

$$= \frac{1}{b}\left[0 + b^2\right]$$

$$= b$$

2. Probability that the received power is greater than average power is

$$F_P(b \le P \le \infty) = \int_{u=b}^{\infty} \frac{1}{b} e^{\frac{-u}{b}}$$

$$= \frac{1}{b}\left[\frac{e^{\frac{-u}{b}}}{-\frac{1}{b}}\right]_b^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1}\right]$$

$$= e^{-1}$$

$$= 0.367$$

Question. 24: 'X' is an exponentional random variable with variance Var(X)=25.

1. What is the PDF of 'X'?
2. Find $E[X^2]$ and
3. Find $P(X > 5)$.

Solution:

We know that, the mean, mean-square and variance for exponentional r.v is

$E[X] = m_1 = b$; $\quad E[X^2] = m_2 = 2b^2$; $\quad Var(X) = \mu_2 = \sigma_X^2 = b^2$

$\therefore b^2 = 25 \Rightarrow b = 5$

1. The PDF function

$$f_X(x) = \begin{cases} \frac{1}{5} e^{\frac{-x}{5}}; & x \ge 0 \\ 0; & x < 0 \end{cases}$$

2. $Var(X) = m_2 - m_1^2 \Rightarrow 25 = m_2 - 25 \Rightarrow m_2 = 50$ $\quad \therefore E[X^2] = 50$

3. $P(X > 5) = \int_{x=5}^{\infty} f_X(x)dx = \int_{x=5}^{\infty} \frac{1}{5} e^{\frac{-x}{b}} = \frac{1}{5}\left[\frac{e^{\frac{-x}{5}}}{-\frac{1}{5}}\right]_5^{\infty}$

$$= -\left[e^{-\infty} - e^{-1}\right] = e^{-1} = 0.367$$

2.5.3 Rayleigh PDF

An Rayleigh distribution function (PDF) can be defined for a continuous random variable X is

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}; & x \ge a \\ 0; & x < a \end{cases}$$

The CDF function is

$$F_X(x) = \int_{u=-\infty}^{x} f_X(u)\,du$$

$$= \int_{u=a}^{x} \frac{2}{b}(u-a)e^{\frac{-(u-a)^2}{b}}\,du$$

Let $\frac{(u-a)^2}{2} = t$

$$\Rightarrow u - a = \sqrt{bt}$$

$$\Rightarrow du = \frac{\sqrt{b}}{2} t^{-1/2} dt$$

If $u = a \Rightarrow t = 0$;

If $u = x \Rightarrow t = \frac{(x-a)^2}{2}$

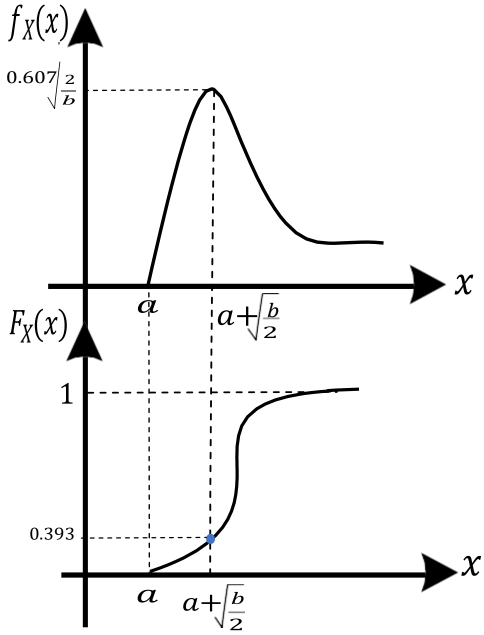
$$= \int_{u=0}^{\frac{(x-a)^2}{b}} \frac{2}{b}\sqrt{bt}\,e^{-t}\,\frac{\sqrt{b}}{2} t^{-1/2} dt$$

$$= \int_{u=0}^{\frac{(x-a)^2}{b}} e^{-t} dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{\frac{(x-a)^2}{b}}$$

$$= -\left[e^{-\frac{(x-a)^2}{2}} - e^{-0}\right]$$

$$= 1 - e^{-\frac{(x-a)^2}{b}}$$



$$\therefore F_X(x) = \begin{cases} 1 - e^{\frac{-(x-a)^2}{b}}; & x \ge a \\ 0; & x < a \end{cases}$$

Applicationas:

- The Rayleigh PDF describes the envelope of one type of noise when passed through a band pass filter.
- It is used to analyze different types of errors in various measurement systems.
- It is useful in describing the noise in RADAR system.
- In communication system, the signal amplitude values of a randomly received signal usually can be modelled as a Rayleigh distribution.

Question. 25: The life time of a computer is expressed in weeks is Rayleigh r.v, its PDF is

$$f_X(x) = \begin{cases} \frac{x}{100} e^{\frac{-x^2}{400}}; & x \geq 0 \\ 0; & x < a \end{cases}$$

1. Find the probability that the computer will not fail in a full week?
2. What is the probability that the life time of a computer will exceed one year?

Solution:

$$P\{0 \leq x \leq 1\} = \int_{x=0}^{1} \frac{x}{200} e^{\frac{-x^2}{400}} dx$$

let $x^2/400 = t \Rightarrow x = 20\sqrt{t}; dx = 10t^{-1/2}dt$

If $x = 0 \Rightarrow t = 0$; If $x = 1 \Rightarrow t = 1/400$

$$= \int_{x=0}^{1/400} \frac{20\sqrt{t}}{200} e^{-t} 10t^{-1/2} dt$$

$$= \int_{x=0}^{1/400} e^{-t} dt = \left[\frac{e^{-t}}{-1}\right]_0^{1/400}$$

$$= -\left[e^{-1/400} - e^{-0}\right] = \left[e^{-0} - e^{-1/400}\right]$$

$= 1 - 1.0025$

$= 0.0025$

2.5.3.1 Statistical Parameters for Rayleigh PDF

Question. 25: Calculate all statistical averages of continuoues random vaiable 'X' with Rayleigh PDF.

Case I.

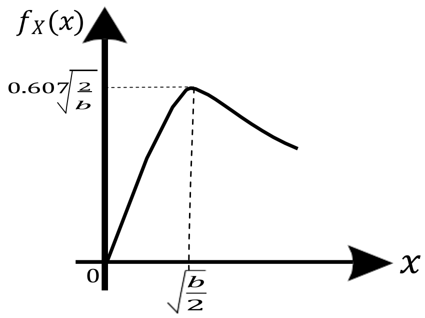
$$f_X(x) = \begin{cases} \frac{2}{b} x e^{-\frac{x^2}{b}}; & x \geq 0 \\ 0; & \text{Else where} \end{cases}$$

Case II.

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) e^{-\frac{(x-a)^2}{b}}; & x \geq a \\ 0; & x < 0 \end{cases}$$

Case I:

1. Mean Value:



$$E[X] = \int_{x=-\infty}^{\infty} x f_X(x) dx$$

$$m_1 = \int_{x=0}^{\infty} x \frac{2}{b} x e^{-\frac{x^2}{b}} dx$$

$$= \frac{2}{b} \int_{x=0}^{\infty} x^2 e^{-\frac{x^2}{b}} dx$$

let $x^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t}$; $dx = \sqrt{b}\, t^{-1/2}\, dt \Rightarrow dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$m_1 = \frac{2}{b} \int_{t=0}^{\infty} bt\, e^{-t} \cdot \sqrt{b}\frac{1}{2\sqrt{t}}dt$$

$$= \sqrt{b} \int_0^{\infty} t^{1-\frac{1}{2}} e^{-t} dt$$

$$= \sqrt{b} \int_0^{\infty} e^{-t} t^{\frac{1}{2}} dt$$

We know that $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$; $\Gamma(n+1) = n\Gamma(n)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \sqrt{b} \int_0^{\infty} e^{-t} t^{\left(\frac{1}{2}+1\right)-1} dt = \sqrt{b}\,\Gamma\left(\frac{1}{2}+1\right) = \sqrt{b}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{b}}{2}\sqrt{\pi}$$

$$m_1 = \frac{\sqrt{b\pi}}{2}$$

2. Mean Square Value:

$$E[X^2] = \int_{x=-\infty}^{\infty} x^2 f_X(x) dx$$

$$m_2 = \int_{x=0}^{\infty} x^2 \cdot \frac{2}{b} x e^{-\frac{x^2}{b}} dx$$

$$= \frac{2}{b} \int_{x=0}^{\infty} x^3 e^{-\frac{x^2}{b}} dx$$

let $x^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t}$; $\quad dx = \sqrt{b}\,t^{-1/2}\,dt \Rightarrow dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; $\qquad$ If $x = \infty \Rightarrow t = \infty$

$$m_2 = \frac{2}{b}\int_{t=0}^{\infty} (\sqrt{bt})^3 e^{-t} \cdot \sqrt{b}\frac{1}{2\sqrt{t}}dt$$

$$= \frac{2}{b}\int_{t=0}^{\infty} b^{\frac{3}{2}} t^{\frac{3}{2}} b^{\frac{1}{2}} \cdot \frac{1}{2} t^{-\frac{1}{2}} e^{-t}dt$$

$$= \frac{1}{b}\int_{t=0}^{\infty} b^2 t e^{-t}dt = b\int_{t=0}^{\infty} t e^{-t}dt$$

$$= b\left[-te^{-t} - \frac{e^{-t}}{(-1)(-1)}\right]_0^{\infty} = b\left[0 - 0 - (0 - 1)\right]$$

$\therefore m_2 = b$

3. 3rd Momentum:

$$E[X^3] = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx$$

$$m_3 = \int_{x=0}^{\infty} x^3 \cdot \frac{2}{b} x e^{-\frac{x^2}{b}}dx$$

$$= \frac{2}{b}\int_{x=0}^{\infty} x^4 e^{-\frac{x^2}{b}}dx$$

let $x^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t}$; $\quad dx = \sqrt{b}\,t^{-1/2}dt \Rightarrow dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; $\qquad$ If $x = \infty \Rightarrow t = \infty$

$$= \frac{2}{b}\int_{t=0}^{\infty} (bt)^2 e^{-t} \cdot \sqrt{b}\frac{1}{2\sqrt{t}}dt$$

$$= b\sqrt{b}\int_0^{\infty} t^{\frac{3}{2}} \cdot e^{-t}dt$$

We know that $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$; $\quad \Gamma(n+1) = n\Gamma(n)$; $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= b\sqrt{b}\int_0^{\infty} e^{-t} t^{\left(\frac{3}{2}+1\right)-1}dt = b\sqrt{b}\,\Gamma\left(\frac{3}{2}+1\right) = b\sqrt{b}\,\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= b\sqrt{b}\,\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}b\sqrt{b}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}b\sqrt{b}\sqrt{\pi}$$

$$m_3 = \frac{3}{4}b\sqrt{b\pi}$$

4. Variance:

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2$$

$$\sigma_X^2 = b - \left(\frac{\sqrt{b\pi}}{2}\right)^2$$

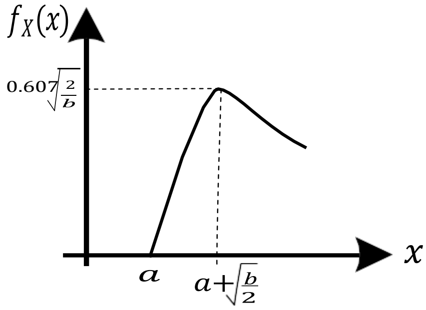
$$= b - \frac{b\pi}{4}$$
$$= b\left(1 - \frac{\pi}{4}\right)$$

5. Standard Deviation: $\sigma_X = \sqrt{b\left(1 - \frac{\pi}{4}\right)}$
6. Skew:

$$\mu_3 = m_3 - 3m_1\mu_2 - m_1^3$$
$$= \frac{3b\sqrt{b\pi}}{4} - 3\sqrt{\frac{b\pi}{2}} \cdot b\left(1 - \frac{\pi}{4}\right) - \left(\frac{\sqrt{b\pi}}{2}\right)^3$$
$$= \frac{3b\sqrt{b\pi}}{4} - \frac{3b\sqrt{b\pi}}{2} + \frac{3b\sqrt{b\pi}\cdot\pi}{8} - \frac{b\pi\sqrt{b\pi}}{8}$$
$$= -\frac{3}{4}b\sqrt{b\pi} + \frac{1}{4}b\pi\sqrt{b\pi}$$
$$\mu_3 = \frac{b\sqrt{b\pi}}{4}(\pi - 3)$$

Case II: Given PDF

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}; & x \geq a \\ 0; & x < 0 \end{cases}$$



1. Mean Value:

$$E[X] = \int_{x=-\infty}^{\infty} x f_X(x)dx$$
$$m_1 = \int_{x=a}^{\infty} x\frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}dx$$
$$= \frac{2}{b}\int_{x=a}^{\infty} x(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

let $(x-a)^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t} + a;\quad dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = a \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$m_1 = \frac{2}{b}\int_{t=0}^{\infty}(\sqrt{bt} + a)\cdot\sqrt{bt}\,e^{-t}\cdot\frac{\sqrt{b}}{2\sqrt{t}}dt$$
$$= \int_0^{\infty}\left(\sqrt{b}\,t^{\frac{1}{2}} + a\right)e^{-t}dt = \int_0^{\infty}\sqrt{b}\,t^{\frac{1}{2}}e^{-t}dt + \int_0^{\infty}ae^{-t}dt$$
$$= \sqrt{b}\int_0^{\infty}e^{-t}\,t^{\left(\frac{1}{2}+1-1\right)}dt + a\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

We know that $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx;\quad \Gamma(n+1) = n\Gamma(n);\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \sqrt{b}\,\Gamma\left(\frac{1}{2}+1\right) + a\left(0+1\right)$$

$$= \sqrt{b}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) + a = \sqrt{b}\,\frac{1}{2}\sqrt{\pi} + a$$

$$m_1 = a + \frac{\sqrt{b\pi}}{2}$$

2. Mean Square Value:

$$E[X^2] = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx$$

$$m_2 = \int_{x=a}^{\infty} x^2 \cdot \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

$$= \frac{2}{b}\int_{x=a}^{\infty} x^2(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

let $(x-a)^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t}+a; \quad dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = a \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$m_2 = \frac{2}{b}\int_{t=0}^{\infty}\left(\sqrt{b}\sqrt{t}+a\right)^2 \cdot e^{-t}\sqrt{b}\sqrt{t}\cdot\sqrt{b}\frac{1}{2\sqrt{t}}dt$$

$$= \int_{t=0}^{\infty}\left(bt + a^2 + 2a\sqrt{b}\sqrt{t}\right)e^{-t}dt$$

$$= b\int_{t=0}^{\infty} te^{-t}dt + a^2\int_{t=0}^{\infty} e^{-t}dt + 2a\sqrt{b}\int_{t=0}^{\infty} e^{-t}t^{\frac{1}{2}}dt$$

$$= b(1) + a^2(1) + 2a\sqrt{b}\,\Gamma\left(\frac{1}{2}+1\right) \qquad \because \Gamma(n+1) = n\Gamma(n)$$

$$= b + a^2 + 2a\sqrt{b}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore m_2 = a^2 + a\sqrt{b\pi} + b$$

3. 3rd Momentum:

$$E[X^3] = \int_{x=-\infty}^{\infty} x^3 f_X(x)dx$$

$$m_3 = \int_{x=0}^{\infty} x^3 \cdot \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

$$= \frac{2}{b}\int_{x=0}^{\infty} x^3(x-a)e^{-\frac{(x-a)^2}{b}}dx$$

let $(x-a)^2/b = t \Rightarrow x = \sqrt{b}\sqrt{t}+a; \quad dx = \sqrt{b}\frac{1}{2\sqrt{t}}dt$

If $x = a \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$= \frac{2}{b}\int_{t=0}^{\infty}\left(\sqrt{b}\sqrt{t}+a\right)^3 \cdot e^{-t}\sqrt{b}\sqrt{t}\cdot\sqrt{b}\frac{1}{2\sqrt{t}}dt$$

$$= \int_{t=0}^{\infty}\left(\sqrt{b}\sqrt{t}+a\right)^3 \cdot e^{-t}dt$$

$$= \int_{t=0}^{\infty}\left[\left(\sqrt{bt}\right)^3 + a^3 + 3bta + 3\sqrt{bt}\,a^2\right]\cdot e^{-t}dt$$

$$= b^{\frac{3}{2}}\int_0^\infty e^{-t}t^{\frac{3}{2}}dt + a^3\int_0^\infty e^{-t}dt + 3ab\int_0^\infty e^{-t}t\,dt + 3a^2\sqrt{b}\int_0^\infty e^{-t}t^{\frac{1}{2}}dt$$

$$= b^{\frac{3}{2}}\int_0^\infty e^{-t}t^{\left(\frac{3}{2}+1\right)-1}dt + a^3(1) + 3ab(1) + 3a^2\sqrt{b}\int_0^\infty e^{-t}t^{\left(\frac{1}{2}+1\right)-1}dt$$

$$= b^{\frac{3}{2}}\,\Gamma\left(\frac{3}{2}+1\right) + a^3 + 3ab + 3a^2\sqrt{b}\,\Gamma\left(\frac{1}{2}+1\right) \qquad \because \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$= b^{\frac{3}{2}}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) + a^3 + 3ab + 3a^2\sqrt{b}\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \because \Gamma(n+1) = n\Gamma(n)$$

$$= b^{\frac{3}{2}}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) + a^3 + 3ab + \frac{3}{2}a^2\sqrt{b}\sqrt{\pi}$$

$$= b^{\frac{3}{2}}\cdot\frac{3}{4}\sqrt{\pi} + a^3 + 3ab + \frac{3}{2}a^2\sqrt{b}\sqrt{\pi} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$m_3 = a^3 + \frac{3b\sqrt{b}\sqrt{\pi}}{4} + \frac{3a^2\sqrt{b}\sqrt{\pi}}{2} + 3ab$$

4. Variance:

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2$$

$$\sigma_X^2 = a^2 + a\sqrt{b}\sqrt{\pi} + b - \left(a + \frac{\sqrt{b\pi}}{2}\right)^2$$

$$= a^2 + a\sqrt{b}\sqrt{\pi} + b - a^2 - \frac{b\pi}{4} - \frac{2a\sqrt{b\pi}}{2}$$

$$= b - \frac{b\pi}{4}$$

$$= b\left(1 - \frac{\pi}{4}\right)$$

5. Standard Deviation: $\sigma_X = \sqrt{b\left(1 - \frac{\pi}{4}\right)}$

6. Skew:

$$\mu_3 = m_3 - 3m_1\mu_2 - m_1^3$$

$$= \left[a^3 + \frac{3b\sqrt{b}\sqrt{\pi}}{4} + \frac{3a^2\sqrt{b}\sqrt{\pi}}{2} + 3ab\right] - 3\left[a + \frac{\sqrt{b\pi}}{2}\right]\cdot\left[b - \frac{b\pi}{4}\right] - \left[b - \frac{b\pi}{4}\right]^3$$

$$= \left[a^3 + 3ab + \frac{3}{2}a^2\sqrt{b\pi} + \frac{3}{4}b\sqrt{b}\sqrt{\pi} - \frac{3}{2}b\sqrt{b}\sqrt{\pi} - \frac{3}{2}b\sqrt{b}\cdot\sqrt{\pi}\sqrt{\pi} + 3ab - \frac{3}{4}ab\pi\right]$$

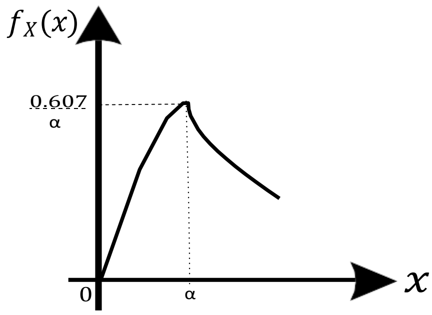
$$- \left[\frac{b\sqrt{b}\cdot\pi\sqrt{\pi}}{8} + a^3 + \frac{3}{4}b\pi a + \frac{3}{2}\sqrt{b\pi}\,a^2\right]$$

$$= \frac{3}{4}b\sqrt{b\pi} - \frac{3}{2}b\sqrt{b\pi} + \frac{3}{8}b\pi\sqrt{b\pi} - \frac{b\pi\sqrt{b\pi}}{8}$$

$$= -\frac{3b\sqrt{b\pi}}{4} + \frac{b\pi\sqrt{b\pi}}{4}$$

$$\mu_3 = \frac{b\sqrt{b\pi}}{4}\pi - 3$$

Question. 26: Calculate all statistical averages of continuoues random vaiable 'X' for given PDF.



Solution: $$f_X(x) = \begin{cases} \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

1. Mean Value:

$$E[X] = \int_{x=-\infty}^{\infty} x f_X(x) dx$$

$$m_1 = \int_{x=0}^{\infty} x \cdot \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \frac{1}{\alpha^2} \int_{x=0}^{\infty} x^2 e^{-\frac{x^2}{2\alpha^2}} dx$$

let $x^2/2\alpha^2 = t \Rightarrow x = \sqrt{2}\alpha\sqrt{t}$

$2xdx = 2\alpha^2 dt \Rightarrow dx = \frac{\alpha^2}{x}dt \Rightarrow dx = \frac{\alpha^2}{\sqrt{2}\alpha\sqrt{t}}dt \Rightarrow dx = \frac{\alpha}{\sqrt{2}\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$m_1 = \frac{1}{\alpha^2}\int_{t=0}^{\infty} 2\alpha^2 t\, e^{-t} \cdot \frac{\alpha}{\sqrt{2}\sqrt{t}} dt$$

$$= \int_{t=0}^{\infty} \sqrt{2}\,\alpha\, t^{\frac{1}{2}} e^{-t} dt$$

$$= \sqrt{2}\,\alpha \int_0^{\infty} e^{-t} \cdot t^{(\frac{1}{2}+1)-1} dt$$

We know that $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$; $\Gamma(n+1) = n\Gamma(n)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \sqrt{2}\,\alpha\,\Gamma\left(\frac{1}{2}+1\right) = \sqrt{2}\,\alpha\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\alpha}{\sqrt{2}}\sqrt{\pi}$$

$$m_1 = \alpha\sqrt{\frac{\pi}{2}}$$

2. Mean Square Value:

$$E[X^2] = \int_{x=-\infty}^{\infty} x^2 f_X(x) dx$$

$$m_2 = \int_{x=0}^{\infty} x^2 \cdot \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \frac{1}{\alpha^2}\int_{x=0}^{\infty} x^3 e^{-\frac{x^2}{2\alpha^2}}\,dx$$

let $x^2/2\alpha^2 = t \Rightarrow x = \sqrt{2}\alpha\sqrt{t}$

$2x\,dx = 2\alpha^2\,dt \Rightarrow dx = \frac{\alpha^2}{x}dt \Rightarrow dx = \frac{\alpha^2}{\sqrt{2}\,\alpha\sqrt{t}}dt \Rightarrow dx = \frac{\alpha}{\sqrt{2}\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$m_2 = \frac{1}{\alpha^2}\int_{t=0}^{\infty} (\sqrt{2t}\,\alpha)^3\, e^{-t} \cdot \frac{\alpha}{\sqrt{2}\sqrt{t}}\,dt$$

$$= \frac{1}{\alpha^2}\int_{0}^{\infty} 2^{\frac{3}{2}}\, t^{\frac{3}{2}}\, \alpha^3 \cdot e^{-t} \cdot \frac{\alpha}{\sqrt{2}\sqrt{t}}\,dt$$

$$= \int_{0}^{\infty} 2^{\frac{3}{2}-\frac{1}{2}}\, t^{\frac{3}{2}-\frac{1}{2}}\, \alpha^2\, e^{-t}\,dt$$

$$= 2\alpha^2\int_{0}^{\infty} t e^{-t}\,dt$$

$$= 2\alpha^2(1)$$

$$m_2 = 2\alpha^2$$

3. 3rd Momentum:

$$E[X^3] = \int_{x=-\infty}^{\infty} x^3 f_X(x)\,dx$$

$$m_3 = \int_{x=0}^{\infty} x^3 \cdot \frac{x}{\alpha^2}\, e^{-\frac{x^2}{2\alpha^2}}\,dx$$

$$= \frac{1}{\alpha^2}\int_{x=0}^{\infty} x^4 e^{-\frac{x^2}{2\alpha^2}}\,dx$$

let $x^2/2\alpha^2 = t \Rightarrow x = \sqrt{2}\alpha\sqrt{t}$

$2x\,dx = 2\alpha^2\,dt \Rightarrow dx = \frac{\alpha^2}{x}dt \Rightarrow dx = \frac{\alpha^2}{\sqrt{2}\,\alpha\sqrt{t}}dt \Rightarrow dx = \frac{\alpha}{\sqrt{2}\sqrt{t}}dt$

If $x = 0 \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\alpha^2}\int_{t=0}^{\infty} (2t\,\alpha^2)^3\, e^{-t} \cdot \frac{\alpha}{\sqrt{2}\sqrt{t}}\,dt$$

$$= \frac{1}{\alpha^2}\int_{t=0}^{\infty} (4t^2\,\alpha^4)\, e^{-t} \cdot \frac{\alpha}{\sqrt{2}\sqrt{t}}\,dt$$

$$= \int_{t=0}^{\infty} 2^{2-\frac{1}{2}}\, t^{2-\frac{1}{2}}\, \alpha^3\, e^{-t}\,dt$$

$$= 2^{\frac{3}{2}}\,\alpha^3 \int_{0}^{\infty} e^{-t}\, t^{\frac{3}{2}}\,dt$$

$$= 2\sqrt{2}\,\alpha^3 \int_{0}^{\infty} e^{-t}\, t^{(\frac{3}{2}+1)-1}\,dt \qquad \because \Gamma(n) = \int_{0}^{\infty} e^{-x}x^{n-1}\,dx$$

$$= 2\sqrt{2}\,\alpha^3\,\Gamma\left(\frac{3}{2}+1\right) \qquad \because \Gamma(n+1) = n\Gamma(n);\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= 2\sqrt{2}\,\alpha^3 \cdot \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) \qquad \because \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$= 2\sqrt{2}\,\alpha^3 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}$$

$$m_3 = 3\alpha^3\sqrt{\frac{\pi}{2}}$$

4. Variance:

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2$$

$$\sigma_X^2 = 2\alpha^2 - \left(\alpha\sqrt{\frac{\pi}{2}}\right)^2$$

$$= 2\alpha^2 - \alpha^2\frac{\pi}{2}$$

$$= \alpha^2\left(2 - \frac{\pi}{2}\right)$$

5. Standard Deviation: $\sigma_X = \alpha\sqrt{2 - \frac{\pi}{2}}$
6. Skew:

$$\mu_3 = m_3 - 3m_1\mu_2 - m_1^3$$

$$= 3\alpha^3\sqrt{\frac{\pi}{2}} - 3\cdot\alpha\sqrt{\frac{\pi}{2}}\cdot\alpha^2\left(2 - \frac{\pi}{2}\right) - \left(\alpha\sqrt{\frac{\pi}{2}}\right)^3$$

$$= 3\alpha^3\sqrt{\frac{\pi}{2}} - 6\alpha^3\sqrt{\frac{\pi}{2}} + 3\alpha^3\frac{\pi}{2}\sqrt{\frac{\pi}{2}} - \alpha^3\frac{\pi}{2}\sqrt{\frac{\pi}{2}}$$

$$= \alpha^3\sqrt{\frac{\pi}{2}}\left[3 - 6 + 3\frac{\pi}{2} - \frac{\pi}{2}\right]$$

$$\mu_3 = \alpha^3\sqrt{\frac{\pi}{2}}\,(\pi - 3)$$

7. Peak value Calculation for the above PDF:

$$f_X(x) = \frac{x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}$$

$$\frac{d}{dx}f_X(x) = 0 \Rightarrow \frac{d}{dx}\left(\frac{x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}\right) = 0$$

$$\Rightarrow \frac{1}{\alpha^2}\frac{d}{dx}\left(xe^{-\frac{x^2}{2\alpha^2}}\right) = 0 \Rightarrow x\frac{d}{dx}e^{-\frac{x^2}{2\alpha^2}} + e^{-\frac{x^2}{2\alpha^2}}\frac{d}{dx}x = 0$$

$$\Rightarrow x\,e^{-\frac{x^2}{2\alpha^2}}\left(\frac{-2x}{2\alpha^2}\right) + e^{-\frac{x^2}{2\alpha^2}}(1) = 0$$

$$\Rightarrow e^{-\frac{x^2}{2\alpha^2}}\left(-\frac{x^2}{\alpha^2} + 1\right) = 0$$

$$\Rightarrow \frac{x^2}{\alpha^2} = 1 \quad \Rightarrow x = \alpha$$

Substitute this $x = \alpha$ value in $f_X(x)$

$$f_X(x) = \frac{\alpha}{\alpha^2} e^{-\frac{\alpha^2}{2\alpha^2}} = \frac{1}{\alpha} e^{-\frac{1}{2}} = \frac{1}{\alpha}(0.6065)$$

$$\therefore f_X(x) = \frac{0.6065}{\alpha}$$

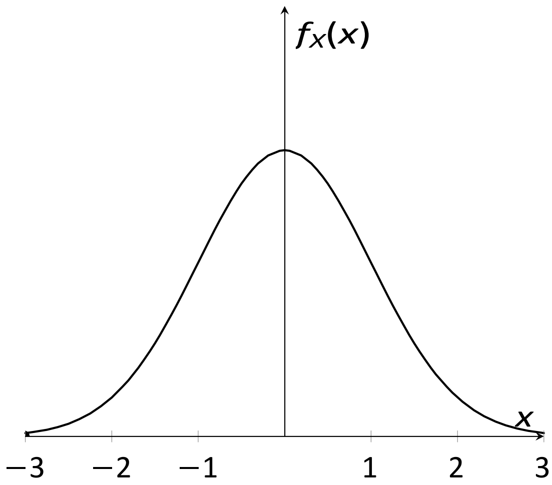
2.5.4 Gaussian (Normal) PDF

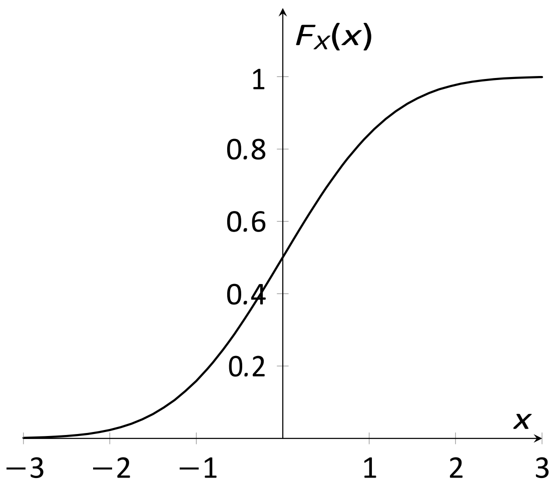
Gaussian PDF is the most important of all PDFs and it enters into nearly all areas of science and engineering. In communication Gaussion is used to represent noise voltage generated across the resistor, shot noise generated in semiconductor devices, thermal noise, noise added by the channel while transmitting information from transmitter to receiver through channel.

Normal PDF: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Normal CDF: $F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}$ (or)

Standard PDF: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} Exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$; where $m = \overline{X} = \mu$





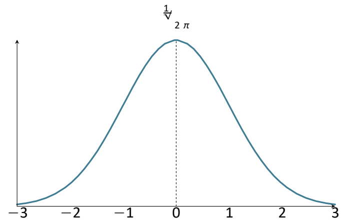
Question. 25: Calculate all statistical averages of continuoues random vaiable 'X' with Gaussion PDF.

1. $$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; & x \geq 0 \\ 0; & \text{Else where} \end{cases}$$

2. $$f_X(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}; & x \geq a \\ 0; & x < 0 \end{cases}$$

Case-1: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

1. Mean Value:



$$E[X] = \int_{x=-\infty}^{\infty} x f_X(x) dx$$

$$m_1 = \int_{x=-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

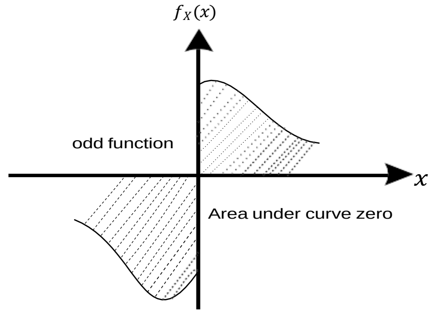
$$= \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} x \cdot e^{-\frac{x^2}{2}} dx$$

This integration of odd function is zero.

$$i.e., \int_{-a}^{a} f_X(x) dx = 0$$

for odd function $f(-x) = -f(x)$;

for even function $f(-x) = f(x)$;



$$\therefore m_1 = E[X] = \overline{X} = 0$$

2. Mean Square Value:

$$E[X^2] = \int_{x=-\infty}^{\infty} x^2 f_X(x) dx$$

$$m_2 = \int_{x=-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

let $x^2/2 = t \Rightarrow x = \sqrt{2}\sqrt{t}$

$2x\, dx = 2\, dt \Rightarrow dx = \frac{1}{x} dt \Rightarrow dx = \frac{1}{\sqrt{2}\sqrt{t}} dt$

If $x = 0$ then $t = 0$; If $x = \infty$ then $t = \infty$

$$= \frac{2}{\sqrt{2\pi}} \int_{t=0}^{\infty} 2t\, e^{-t} \cdot \frac{1}{\sqrt{2}\sqrt{t}} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{t=0}^{\infty} e^{-t}\, t^{\frac{1}{2}} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{t=0}^{\infty} e^{-t} \cdot t^{(\frac{1}{2}+1)-1} dt$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

$$\therefore\ m_2 = E[X^2] = \overline{X^2} = 1$$

3. 3rd Momentum about origin:

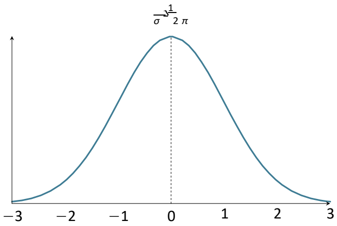
$$E[X^3] = m_3 = \int_{-\infty}^{\infty} x^3 f_X(x)dx = 0 \qquad \because \text{ it is odd function}$$

4. Variance: $\sigma_X^2 = m_2 - m_1^2 = 1 - 0^2 = 1$
5. Standard deviation: $\sigma_X = 1$
6. Skew: $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 0 - 0 - 0 = 0$

Case-2: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} Exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$

1. Mean Value:

$$E[X] = \int_{x=-\infty}^{\infty} x f_X(x)dx$$



$$m_1 = \int_{x=-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

let $x - m = y \Rightarrow x = y + m \Rightarrow dx = dy$

If $x = -\infty \Rightarrow y = -\infty$; If $x = \infty \Rightarrow y = \infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (y+m)\, e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{-\frac{y^2}{2\sigma^2}}}_{=0} + \frac{m}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} m\, e^{-\frac{y^2}{2\sigma^2}}$$

$$= \frac{m}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{-\frac{y^2}{2\sigma^2}} \qquad \because \text{ odd function}$$

let $\frac{y^2}{2\sigma^2} = t \Rightarrow y = \sqrt{2}\,\sigma\sqrt{t}$

$$2y\,dy = 2\sigma^2 dt \Rightarrow dy = \frac{\sigma^2}{y} dt \Rightarrow dy = \frac{\sigma^2}{\sqrt{2}\,\sigma\sqrt{t}} dt \Rightarrow dy = \frac{\sigma}{\sqrt{2}} t^{-\frac{1}{2}} dt$$

$$= \frac{m}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} \cdot \frac{\sigma}{\sqrt{2}} t^{-\frac{1}{2}} dt$$

$$= \frac{m}{2\sqrt{\pi}} \int_{t=-\infty}^{\infty} e^{-t} \cdot t^{-\frac{1}{2}} dt$$

$$= 2 \cdot \frac{m}{2\sqrt{\pi}} \int_{t=0}^{\infty} e^{-t} \cdot t^{\frac{1}{2}-1} dt \qquad \because \text{ even function}$$

$$= \frac{m}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{m}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

$E[X] = m$

2. Mean Squre Value:

$$E[X^2] = \int_{x=-\infty}^{\infty} x^2 f_X(x)dx$$

$$m_1 = \int_{x=-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{x=0}^{\infty} x^2 \cdot e^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad \because \text{ even function}$$

let $\frac{x-m}{\sigma} = y \Rightarrow x = \sigma y + m \Rightarrow dx = \sigma\, dy$

If $x = 0 \Rightarrow y = 0$; If $x = \infty \Rightarrow y = \infty$

$$= \frac{2}{\not\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + m)^2\, e^{-\frac{y^2}{2\sigma^2}} \not\sigma\, dy$$

$$= \frac{2}{\sqrt{2\pi}} \left[\sigma^2 \int_0^{\infty} y^2 e^{-\frac{y^2}{2}} dy + m^2 \int_0^{\infty} e^{-\frac{y^2}{2}} dy + 2\sigma m \int_0^{\infty} e^{-\frac{y^2}{2}} dy^{=0}\right]$$

$$= \sigma^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} y^2 e^{-\frac{y^2}{2}} dy + m^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{y^2}{2}} dy \qquad \because \text{ even function}$$

let $\frac{y^2}{2} = t \Rightarrow y = \sqrt{2t}$

$2y\, dy = 2\, dt \Rightarrow dy = \frac{1}{\sqrt{2t}} dt$

If $y = 0$ then $t = 0$; If $y = \infty$ then $t = \infty$;

$$= \sigma^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} 2t \cdot e^{-t} \frac{1}{\sqrt{2t}} dt + m^2\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} \frac{1}{\sqrt{2t}} dt$$

$$= \sigma^2 \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot t^{(\frac{1}{2}+1)-1} dt + \frac{m^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot t^{\frac{1}{2}-1} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right) + \frac{m^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \qquad \because \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) + \frac{m^2}{\sqrt{\pi}} \sqrt{\pi} \qquad \because \Gamma(n+1) = n\Gamma(n); \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} + m^2$$

$$E[X^2] = \sigma^2 + m^2$$

3. Variance: σ_X^2 or μ_2

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2$$

$$= \sigma^2 + m^2 - m^2$$

$$\therefore \mu_2 = \sigma^2$$

4. Standard Deviation: $\sigma_X = \sigma$

5. Skew: $\mu_3 = 0$ $\because$ It is symmetry about mean $\overline{X}$

2.5.4.1 Q-function:

The Gaussion function can be defined as

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{X})^2}{2\sigma^2}} \tag{2.3}$$

If $\overline{X} = 0$ and $\sigma = \pm 1$ then $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. This is called "Normal Gaussion PDF function".

The Normal Gaussion CDF function is

$$F_X(x) = P(-\infty \le X \le x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{2.4}$$

This integral can not be evaluated in closed form and it must be computed numerically. It is convenient to use the function $Q(\cdot)$, defined as

$$Q(x) = P(X > x) = \int_{u=x}^{\infty} f_X(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{u=x}^{\infty} e^{-\frac{u^2}{2}} du$$

The area under $f_X(x)$ from 0 to ∞ is $Q(\cdot)$. From the symmetry of $f_X(x)$ about origion and total area under $f_X(x)$ is 1.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \Rightarrow \int_{-\infty}^{0} f_X(x)\, dx + \int_{0}^{\infty} f_X(x)\, dx = 1 \Rightarrow F_X(x) + Q(x) = 1$$

$\therefore F_X(x) = 1 - Q(x)$

$$P\{X < x\} = F_X(x) = 1 - Q(x)$$
$$P\{X > x\} = Q(x)$$

From equation (2.3), Gaussion PDF of $F_X(x)$ is

$$F_X(x) = P(-\infty \le X \le x) = P(X \le x)$$
$$= \int_{-\infty}^{x} f_X(u)\, du = \frac{1}{\sigma_X\sqrt{2\pi}} \int_{u=-\infty}^{x} e^{-\frac{(u-\overline{X})^2}{2\sigma_X^2}} du$$

let $Z = \dfrac{u-\overline{X}}{\sigma_X} \Rightarrow du = \sigma_X\, dZ$

If $u = -\infty \Rightarrow Z = -\infty;$ $u = x \Rightarrow Z = \dfrac{x-\overline{X}}{\sigma_X}$

$$= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{Z=-\infty}^{\frac{x-\overline{X}}{\sigma_X}} e^{-\frac{Z^2}{2}} \cdot \sigma_X\, dZ$$

$$= \frac{1}{\sqrt{2\pi}} \int_{Z=-\infty}^{\frac{x-\bar{X}}{\sigma_X}} e^{-\frac{Z^2}{2}}\, dZ$$

$$= P\{X < x\}$$

$$\therefore F_X(x) = 1 - Q\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

$$P\{X < x\} = F_X\left(\frac{x-\bar{X}}{\sigma_X}\right) = 1 - Q\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

$$P\{X > x\} = 1 - F_X\left(\frac{x-\bar{X}}{\sigma_X}\right) = Q\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

Summary:

1. Q(·) Definition:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{u=x}^{\infty} e^{-\frac{u^2}{2}}\, du$$

2. Property

$$Q(-x) = 1 - Q(x)$$

3. Simple Upper Bound

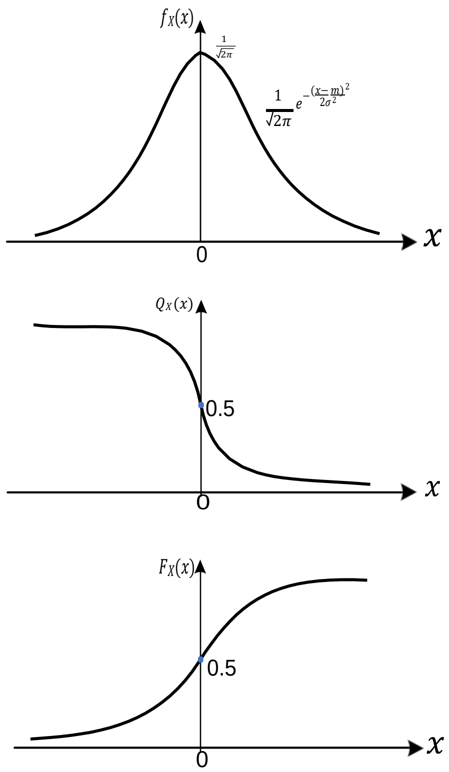
$$Q(x) < \frac{1}{2} e^{-\frac{x^2}{2}}$$

4. Relation to Error Functions

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right),$$

$$erfc(x) = 2Q(x\sqrt{2})$$

5. Good Approximation (good for programming in calculator)



$$Q(x) \approx \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

where $a = \frac{1}{\pi}$, $b = 2\pi$.

Question. 26: Find the probability of the event $X \leq 5.5$ for Gausssion random variable having $m = 3$ and $\sigma_X = 2$.

Solution:

$$P(x \leq 5.5) = P(-\infty \leq X \leq 5.5) = F_X(x) = 1 - Q\left(\frac{x - \overline{X}}{\sigma_x}\right)$$

$$\frac{x - \overline{X}}{\sigma_X} = \frac{5.5 - 3}{2} = 1.2$$

$$\therefore F_X(x) = 1 - Q(1.25) = F_X(1.25) = 0.8944$$

(or another method)

$$= 1 - Q(1.25)$$

$$= 1 - 0.1056$$

$P(x \leq 5.5) = 0.8944$ and also $P(x \geq 5.5) = 0.1056$

Question. 27: Find the probability of the event $X \leq 7.3$ for Gausssion random variable having $m = 3$ and $\sigma_X = 0.5$.

Solution:

$$P(x \leq 5.5) = P(-\infty \leq X \leq 5.5) = F_X(x) = 1 - Q\left(\frac{x - \overline{X}}{\sigma_x}\right)$$

$$\frac{x - \overline{X}}{\sigma_X} = \frac{7.3 - 7}{0.5} = 1.2$$

$$\therefore F_X(x) = F_X(0.6) = 1 - Q(0.6)$$

$$= 1 - 0.2743$$

$P(x \leq 5.5) = 0.7257$ and also $P(x \geq 5.5) = 0.2743$

Question. 27: Assume that the height of clouds above the ground at some location is a gaussion r.v with $m = 1890\ m$ and $\sigma_X = 460\ m$. Find the probability that the clouds will be higher than $2750\ m$?

Solution:

$$P(X > 2750) = 1 - P(-\infty \leq X \leq 2750) \qquad \because P(X > x) = Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$

$$= 1 - F_X(2750) \qquad \because P(X < x) = 1 - Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$

$$= 1 - \left[1 - Q\left(\frac{x - \overline{X}}{\sigma_X}\right)\right]$$

$$= Q\left(\frac{2750 - 1830}{460}\right) = Q(2.0) = 0.2275 \times 10^{-1}$$

$$\therefore P(X > 2750) = 0.02275$$

Question. 28: An analog signal received at the dector (measured in μV) may be modeled as Gaussion r.v with the mean value 200 and standard deviation 256. What is the probability that the signal is larger than $250\ \mu V$?

Solution:

$$P(X > 250) = 1 - P(-\infty \le X \le 250) \qquad \because P(X > x) = Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$

$$= 1 - F_X(2750) \qquad \because P(X < x) = 1 - Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$

$$= 1 - \left[1 - Q\left(\frac{2750 - 1830}{460}\right)\right]$$

$$= Q(0.195) = 0.4247$$

$$\therefore P(X > 250) = 0.4247$$

2.5.4.2 Error function “erf”:

The Gaussion function PDF can be defined as

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{X})^2}{2\sigma^2}} \qquad (2.5)$$

Let $\overline{X} = 0$ then $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. This is called “Normal Gaussion PDF function”.
Here $\overline{X} = m_1 = 0$ means, in communication noise is zero.
The Normal Gaussion CDF function is

$$F_X(x) = P(-\infty \le X \le x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}} du \qquad (2.6)$$

This integral is not easily evaluated and it can be evaluated using standard function called “error function”, which is defined as

$$\boxed{erf(u) = \frac{2}{\sqrt{\pi}} \int_{u=0}^{x} e^{-u^2} du}$$

The complementary error function

$$erfc(u) = 1 - erf(u)$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_{u=0}^{x} e^{-u^2} du$$

$$\boxed{\therefore erfc(u) = \frac{2}{\sqrt{\pi}} \int_{u=x}^{\infty} e^{-u^2} du}$$

From the equation (2.5), the CDF can be evaluated by

$$F_X(x) = P(-\infty \le X \le x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}} du$$

$$= \int_{u=-\infty}^{\infty} \frac{e^{-\frac{u^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} du - \int_{u=x}^{\infty} \frac{e^{-\frac{u^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} du$$

let $z = \frac{u}{\sqrt{2}\,\sigma} \Rightarrow dz = \frac{du}{\sqrt{2}\,\sigma} \Rightarrow du = \sqrt{2}\,\sigma\, dz$

If $u = x \Rightarrow z = \frac{x}{\sqrt{2}\,\sigma}$; and $z^2 = \frac{x^2}{2\,\sigma^2}$; If $u = \infty \Rightarrow z = \infty$

$$= 1 - \int_{z=\frac{x}{\sqrt{2}\sigma}}^{\infty} \frac{e^{-\frac{z^2 2\sigma^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \sqrt{2}\,\sigma\, dz$$

$$= 1 - \frac{1}{\sqrt{\pi}} \int_{z=\frac{x}{\sqrt{2}\sigma}}^{\infty} e^{-z^2} dz$$

$$= 1 - \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{z=\frac{x}{\sqrt{2}\sigma}}^{\infty} e^{-z^2} dz$$

$$= 1 - \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

$$\therefore F_X(x) = 1 - \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

2.5.4.3 Relationship between Q(·) and erfc(·):

$$erfc\left(\frac{x}{\sqrt{2}\,\sigma}\right) = 2 - 2F_X(x)$$

$$= 2\,[1 - F_X(x)]$$

$$= 2Q(x)$$

$$\therefore Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

(or)

$$Q(x) = 1 - \frac{1}{2} erf\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

Question. 29: ‘X’ is Gaussian r.v with $E[X] = 0$ and $P[\,|X| \le 10\,] = 0.1$. What is the standard deviation σ_X?

Solution:

$$P(|X| \le 10) = P(-10 \le X \le 10) = F_X[10] - F_X[-10]$$

$$= F_X[10] - (1 - F_X[10]) = 2F_X[10] - 1$$

$$= 2Q\left(\frac{10-0}{\sigma_X}\right) - 1 \qquad \because F_X(x) = Q\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

$$= 2Q\left(\frac{10}{\sigma_X}\right) - 1 = 0.1$$

$$\Rightarrow Q\left(\frac{10}{\sigma_X}\right) = 0.55$$

$$\Rightarrow \frac{10}{\sigma_X} = 0.15 \qquad \therefore \sigma_X = 66.6$$

Question. 30: Life time of IC chips manufactured by a semiconcuctor manufacturer is approximately normally distributed with mean = 5×10^6 hours and standard deviation is 5×10^5 hours. A mainframe manufacture requires that at least 95% of a batch should have a life time greater than 4×10^6 hours will the deal be made?

Solution:

$$P(X > 4 \times 10^6) = 1 - P(X < 4 \times 10^6)$$

$$= 1 - Q\left(\frac{4 \times 10^6 - m}{\sigma}\right) \qquad \because P(X \le x) = Q\left(\frac{x-m}{\sigma}\right)$$

$$= 1 - Q\left(\frac{4 \times 10^6 - 5 \times 10^6}{5 \times 10^5}\right)$$

$$= 1 - Q\left(\frac{-10^6}{5 \times 10^5}\right) = 1 - Q(-2) = 1 - Q(2) = 0.0228$$

This deal can be made but with less certinity

Question. 30: The average life of a certain type of electric bulb is 1200 hours. What percentage of this type of bulb is expected to fail in the first 800 hours of working? What percentage of expected to fail between 800 and 1000 hours? Assume normal distribution with $\sigma = 200$ hours.

Solution: (i.)

$$P(X < x) = 1 - Q\left(\frac{x-m}{\sigma}\right)$$

$$= 1 - Q\left(\frac{800-1200}{200}\right) = 1 - Q(-2) = 0.0228$$

$\therefore$ 2.28% of bulbs is expected to fail in first 800 hours of working.

(ii.)

$$P(800 \le X \le 1000) = F_X(1000) - F_X(800)$$

$$= F_X\left(\frac{1000-1200}{200}\right) - F_X\left(\frac{800-1200}{200}\right)$$

$$= F_X(-1) - F_X(-2)$$

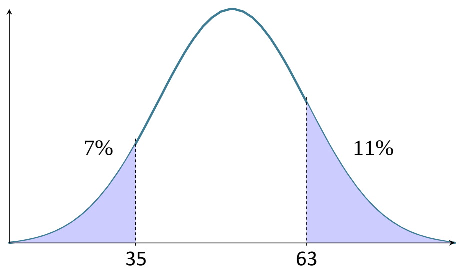
$$= 1 - F_X(1) - [1 - F_X(2)] \qquad \because F_X(x) = Q\left(\frac{x - \overline{X}}{\sigma}\right)$$

$$= Q(1) - Q(2) = 0.1587 - 0.0228 = 0.1359$$

$\therefore$ 13.59% is expected to fail between 800 and 1000 hours.

Question. 31: in a distribution exactly Gaussion 7% of items are under 35 and 11% are over 63. Find the mean and standard deviation of the distribution. Also find how many items are between 40 and 60 out of 200 items?

Solution:



Given $P(X \leq 35) = 0.07$ and $P(X > 63) = 0.11$

We know that

$$P(X \leq x) = P(-\infty \leq X \leq x) = F_X\left(\frac{x - \overline{X}}{\sigma_X}\right) = 1 - Q\left(\frac{x - \overline{X}}{\sigma_X}\right) \qquad (2.7)$$

$$P(X \geq x) = P(x \leq X \leq \infty) = F_X\left(\frac{x - \overline{X}}{\sigma_X}\right) = Q\left(\frac{x - \overline{X}}{\sigma_X}\right) \qquad (2.8)$$

$$P(X \leq 35) = F\left(\frac{35 - \overline{X}}{\sigma_X}\right) = 0.07$$

let $m = \frac{35 - \overline{X}}{\sigma_X}$ value is negative, then $F\{-m\} = 1 - F\{m\}$

$$= 1 - F\left(\frac{35 - \overline{X}}{\sigma_X}\right) = 0.07$$

$$= F\left(\frac{35 - \overline{X}}{\sigma_X}\right) = 0.93$$

From above , check $F[\cdot]$ in table and put a negative sign in front.

As it is the left of mean. i.e., about $m = 0$.

$$\Rightarrow \frac{35 - \overline{X}}{\sigma_X} = -1.48 \qquad \because F_X(1.48) = 0.93$$

$$P(X \leq 35) \Rightarrow \overline{X} - 1.48\,\sigma_X - 35 = 0$$

From equation (2.8),

$$P(X \geq 63) = 1 - P(X \leq 63) = F\left(\frac{363 - \overline{X}}{\sigma_X}\right) = 0.11$$

$$\Rightarrow F\left(\frac{63-\overline{X}}{\sigma_X}\right) = 1 - 0.11 = 0.89$$

let $m = \frac{63-\overline{X}}{\sigma_X}$ value is positive, then $F\{m\} = F\{m\}$

$$\Rightarrow F\left(\frac{63-\overline{X}}{\sigma_X}\right) = 0.89$$

From above , check $F[\cdot]$ in table and put a positive sign in front.

As it is the left of mean. i.e., about $m = 0$.

$$\Rightarrow \frac{63-\overline{X}}{\sigma_X} = 1.23 \qquad \because F_X(1.23) = 0.89$$

$$P(X \geq 63) \Rightarrow \overline{X} + 1.23\,\sigma_X - 63 = 0$$

From the above $P(X \leq 35)$ and $P(X \geq 63)$ solutions, we will get $\sigma_X = 10.3;\ \overline{X} = 50.244$

$$P\{40 \leq X \leq 60\} = F_X[60] - F_X[40]$$

$$= F_X\left(\frac{60-50.244}{10.3}\right) - F_X\left(\frac{40-50.244}{10.3}\right)$$

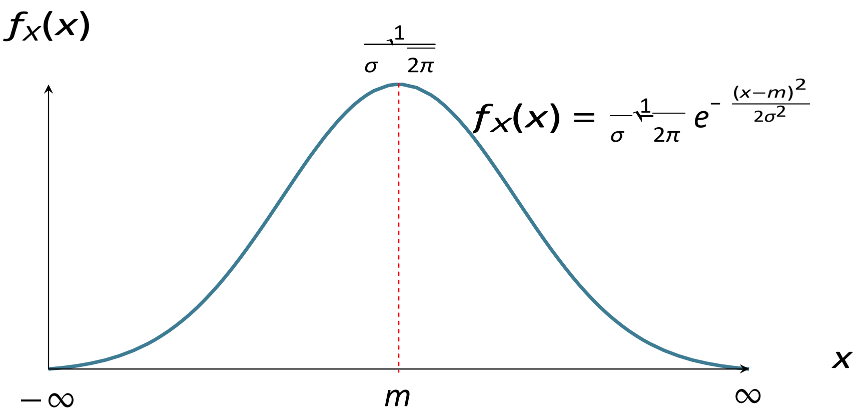
$$= F_X[0.9] - F_X[-0.9]$$

$$= F_X[0.9] - (1 - F_X[0.9])$$

$$= 2F_X[0.9] - 1 = 2 \times 0.8159 - 1$$

$$P\{40 \leq X \leq 60\} = 0.6318$$

2.5.4.4 Properties of Gaussion PDF



1. The Gaussion PDF is used to describe the noise generated by resistor (thremal), noise generated by semiconductor (shot noise) and noise generated by channel (channel transmitter).
2. The Gaussion PDF is symmetrical about its mean and it is bell shaped curve

3. when S.D $\sigma_X = 1$ and mean $m_1 = 0 = \overline{X}$, General Gaussion PDF is called normal gaussion PDF as shown in figure.

4. The maximum value of normal Gaussion PDF is $\frac{1}{\sqrt{2\pi}}$ at $x = 0$.

5. When

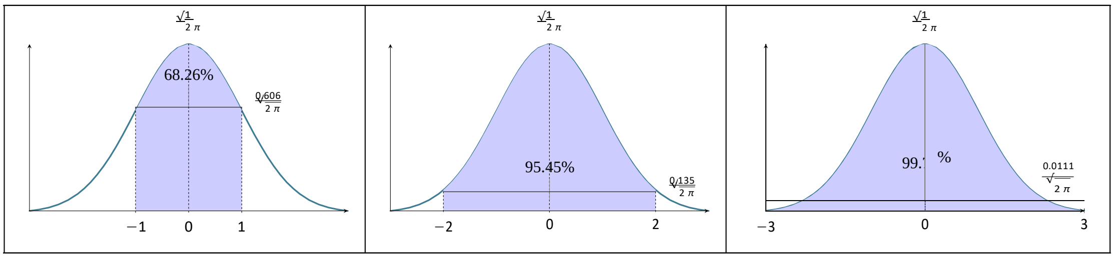
$$x = \pm 0,\ f_X(x) = \frac{1}{\sqrt{2\pi}}$$

$$x = \pm 1,\ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{0.606}{\sqrt{2\pi}}$$

$$x = \pm 2,\ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-2} = \frac{0.135}{\sqrt{2\pi}}$$

$$x = \pm 3,\ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-4.5} = \frac{0.0111}{\sqrt{2\pi}}$$

$\therefore$ $x = \pm 1, \pm 2, \pm 3$, the maximum value falls to $\frac{0.606}{\sqrt{2\pi}}, \frac{0.135}{\sqrt{2\pi}}, \frac{0.0111}{\sqrt{2\pi}}$ respectively as shown in figure.



6. For $x = \pm 1, \pm 2, \pm 3$, the area of the curve is 68.26%, 95.45%, and 99.73% of the total area respectively as shown in Figure.

$$P(-1 \le X \le 1) = F(1) - F(-1) = 2F(1) - 1$$
$$= 2(0.8413) - 1$$
$$= 0.6826 = 68.26\%$$

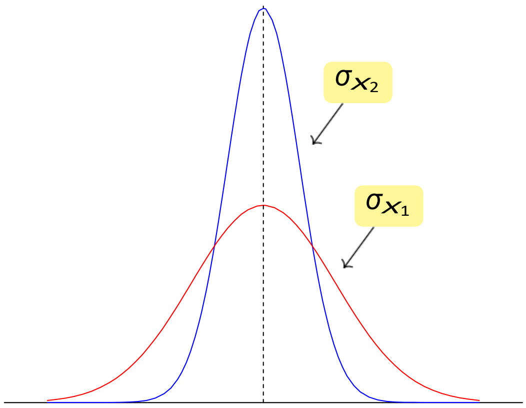
$$P(-2 \le X \le 2) = F(2) - F(-2) = 2F(2) - 1$$
$$= 2(0.9772) - 1$$
$$= 0.9545 = 95.45\%$$

$$P(-3 \le X \le 3) = F(3) - F(-3) = 2F(3) - 1$$
$$= 2(0.9987) - 1$$
$$= 0.9973 = 99.73\%$$

7. A continuous r.v X_X and another continuous r.v X_2, their mean and variance are $\overline{X_1}, \sigma_{X_1}$, and $\overline{X_2}, \sigma_{X_2}$, then

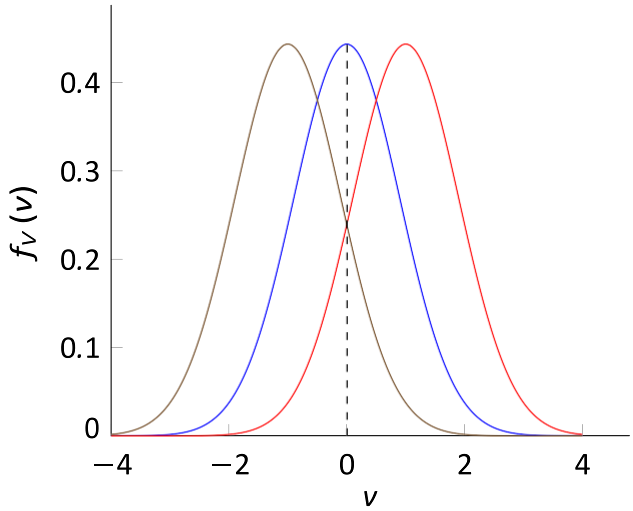
$$f_{X_1}(x) = \frac{1}{\sigma_{X_1}\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2_{X_1}}} \qquad f_{X_1}(x) = \frac{1}{\sigma_{X_2}\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2_{X_2}}}$$

If $\sigma_{X_1} > \sigma_{X_2}$ then the normal distribution PDF is



8. In communication system, the noise representations with respect to different amplitudes is as follows.

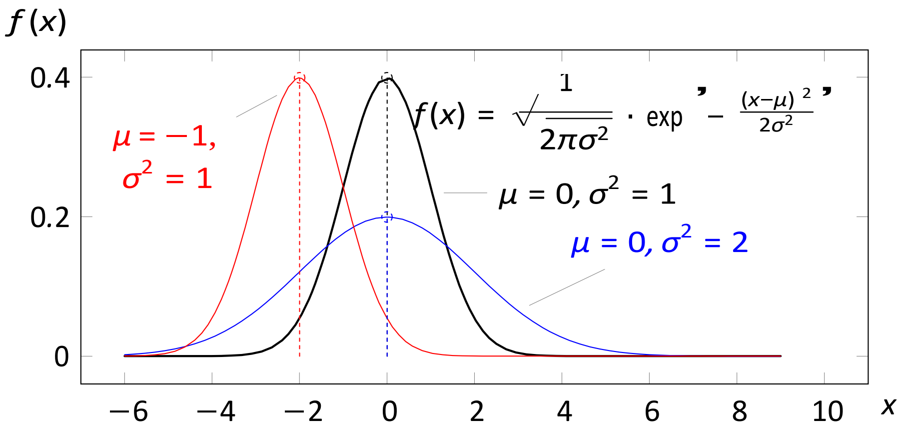
   - If the noise has equal positive amplitude and negative amplitude then its mean is zero as shown in figure.
   - If the noise has more positive amplitude then its mean is positive as shown in figure.
   - If the noise has more negative amplitude then its mean is negative as shown in figure.

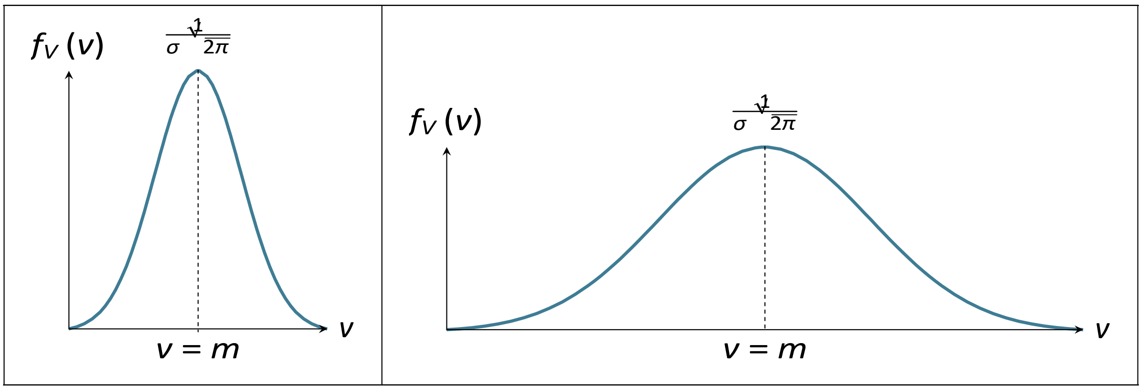


9. The noise voltages with respect to standard deviation are

   - For low value of standard deviation noise voltages are more closed to mean as shown in figure.

- For high standard deviation noise will have more amplitude variations about mean as shown in figure.





x \ Δx	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2.5 Approximations of $F_{0;1}(x + \Delta x)$

$$F_{0;1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t \qquad F_{0;1}(1.65) \approx 0.9505$$

$$F_{\mu;\sigma^2}(x) = F_{0;1}\left(\frac{x-\mu}{\sigma}\right) \qquad F_{0;1}(-x) = 1 - F_{0;1}(x)$$

2.6 Discrete random variable - Statistical parameters

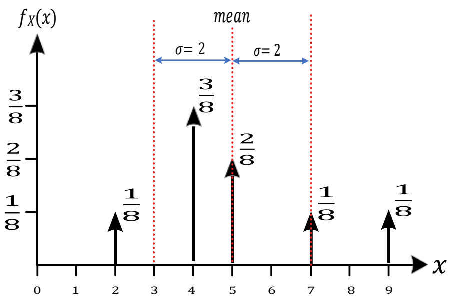
Let 'X' be the discrete r.v which takes the values x_i where $i = -\infty$ to $+\infty$ with probability density function.

1. Expectation	$E[X] = \overline{X} = m_1 = \sum_{i=-\infty}^{\infty} x_i f_X(x_i)$
2. Mean square value	$E[X^2] = \overline{X^2} = m_2 = \sum_{i=-\infty}^{\infty} x_i^2 f_X(x_i)$
3. Third moment	$E[X^3] = m_3 = \sum_{i=-\infty}^{\infty} x_i^3 f_X(x_i)$
4. Variance	$\sigma_X^2 = \mu_2 = Var[X] = m_2 - m_1^2$
5. Standard deviation	$\sigma_X = \sqrt{\mu_2}$
6. Skew deviation	$\mu_3 = m_3 - 3m_1\mu_2 - m_1^3$

Question. 32: Find mean, mean square, variance, and standard deviation of statistical data is 2, 4, 4, 4, 5, 5, 7, 9.

Solution:

x_i	2	4	5	7	9
$f_X(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$



$$E[X] = \sum_x x_i f_X(x)$$

$$= 2 \times \frac{1}{8} + 4 \times \frac{3}{8} + 5 \times \frac{2}{8} + 7 \times \frac{1}{8} + 9 \times \frac{1}{8}$$

$$= \frac{2 + 12 + 10 + 7 + 9}{8} = 5$$

$$E[X^2] = \sum_x x_i^2 f_X(x)$$

$$= 4 \times \frac{1}{8} + 16 \times \frac{3}{8} + 25 \times \frac{2}{8} + 49 \times \frac{1}{8} + 81 \times \frac{1}{8}$$

$$= \frac{4 + 48 + 50 + 49 + 81}{8} = \frac{232}{8} = 29$$

$$E[X^3] = \sum_x x_i^3 f_X(x)$$

$$= 8 \times \frac{1}{8} + 64 \times \frac{3}{8} + 125 \times \frac{2}{8} + 343 \times \frac{1}{8} + 729 \times \frac{1}{8}$$

$$= \frac{8 + 192 + 250 + 343 + 729}{8} = 29$$

Variance: $\sigma_X^2 = m_2 - m_1^2 = 29 - 25 = 4$ (or) other method

$$\sigma_X^2 = E[(x_i - \overline{X})^2] = \sum_x (x_i - \overline{X})^2 f_X(x)$$

$$= \frac{(2-5)^2 + 3(4-5)^2 + 2(5-5)^2 + (7-5)^2 + (9-5)^2}{8}$$

$$= \frac{3^2 + 3 + 2^2 + 4^2}{8} = \frac{9 + 3 + 4 + 16}{8} = 4$$

$\therefore \sigma_X^2 = 4$

standard deviation: $\sigma_X = \sqrt{4} = \pm 2$

skew:

$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 190.25 - 3 \times 5 \times 4 - 5^3 = 190.25 - 60 - 125 = 5.25$

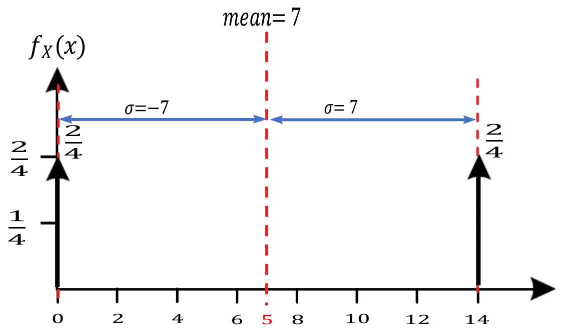
Skewness: $\frac{\mu_3}{\sigma_X^3} = \frac{5.25}{2^3} = 0.65625$

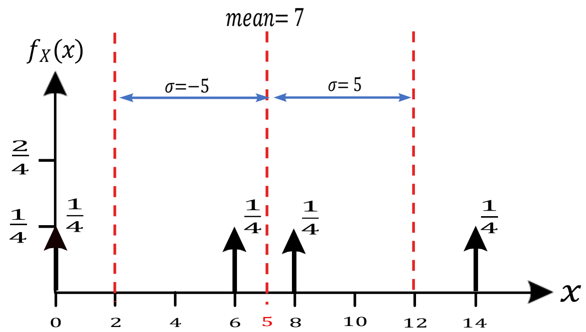
Question. 33: Find all statistical parameters for given statistical data.

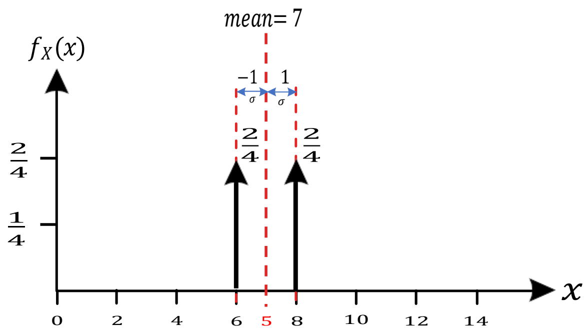
(i) 0, 0, 14, 14 (ii) 0, 6, 8, 14 (iii) 6, 6, 8, 8

Solution:

x_i: 0, 14; $f_X(x)$: $\frac{2}{4}$, $\frac{2}{4}$	x_i: 0, 6, 8, 14; $f_X(x)$: $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$	x_i: 6, 8; $f_X(x)$: $\frac{2}{4}$, $\frac{2}{4}$
$E[X] = 0\ \frac{2}{4} + 14\ \frac{2}{8}$ $= 7$	$E[X] = 0\ \frac{1}{4} + 6\ \frac{1}{4} + 8\ \frac{1}{4} + 14\ \frac{1}{4}$ $= 7$	$E[X] = 6\ \frac{2}{4} + 8\ \frac{2}{4}$ $= 7$
$E[X^2] = 0^2\ \frac{2}{4} + 14^2\ \frac{2}{8}$ $= 0\ \frac{2}{4} + 196\ \frac{2}{4}$ $= 7$	$E[X^2] = 0^2\ \frac{1}{4} + 6^2\ \frac{1}{4} + 8^2\ \frac{1}{4} + 14^2\ \frac{1}{4}$ $= 0\ \frac{1}{4} + 36\ \frac{1}{4} + 64\ \frac{1}{4} + 196\ \frac{1}{4}$ $= 0 + 9 + 16 + 49$ $= 74$	$E[X^2] = 6^2\ \frac{2}{4} + 8^2\ \frac{2}{4}$ $= 36\ \frac{2}{4} + 64\ \frac{2}{4}$ $= 18 + 32$ $= 50$
$E[X^3] = 0^3\ \frac{2}{4} + 14^3\ \frac{2}{8}$ $= 0\ \frac{2}{4} + 2744\ \frac{2}{4}$ $= 1372$	$E[X^3] = 0^3\ \frac{1}{4} + 6^3\ \frac{1}{4} + 8^3\ \frac{1}{4} + 14^3\ \frac{1}{4}$ $= 0\ \frac{1}{4} + 216\ \frac{1}{4} + 512\ \frac{1}{4} + 2744\ \frac{1}{4}$ $= 0 + 54 + 128 + 686$ $= 868$	$E[X^3] = 6^3\ \frac{2}{4} + 8^3\ \frac{2}{4}$ $= 216\ \frac{2}{4} + 512\ \frac{2}{4}$ $= 108 + 256$ $= 364$
$\sigma_X^2 = m_2 - m_1^2 = 98 - 49 = 49$	$\sigma_X^2 = m_2 - m_1^2 = 74 - 49 = 25$	$\sigma_X^2 = m_2 - m_1^2 = 50 - 49 = 1$
S.D $= \sigma_X = \sqrt{49} = \pm 49$	S.D $= \sigma_X = \sqrt{25} = \pm 5$	S.D $= \sigma_X = \sqrt{1} = \pm 1$
Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 1372 - 3 \times 7 \times 49 - 7^3$ $= 1372 - 1029 - 343$ $= 0$	Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 868 - 3 \times 7 \times 28 - 7^3$ $= 0$	Skew $\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3$ $= 364 - 3 \times 7 \times 1 - 7^3$ $= 0$







All three have the same expected value, $E[X] = 7$, but the "spread" in the distributions is quite different. Variance is a formal quantification of "spread". There is more than one way to quantify spread; variance uses the average square distance from the mean.

Standard deviation is the square root of variance: $SD(X) = \sqrt{Var(X)}$. Intuitively, standard deviation is a kind of average distance of a sample to the mean. (Specifically, it is a root-meansquare [RMS] average.) Variance is the square of this average distance.

NOTE:

1. The variance and standard deviation are closely related, measures spreding of data about mean value.

2. Standard deviation is propotional to width of PDF $f_X(x)$

3. For low standard deviations data and events are more closely existed about mean and viceversa.

Question. 34: A fair coin is tossed three times. Let 'X' be the number of tails appearing. Find PDF, CDF and also its statistical parameters ?

Solution:

Sample space S	=	{ HHH	HHT	HTH	THH	HTT	THT	TTH	TTT }
D.r.v 'X'	=	{ x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8 }
Getting Tail	=	{ 0	1	1	1	2	2	2	3 }

$$P(X = 0) = P(x_1) = \frac{1}{8}$$

$$P(X = 1) = P(x_2) + P(x_3) + P(x_4) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 1) = P(x_5) + P(x_6) + P(x_7) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X = 3) = P(x_8) = \frac{1}{8}$$

x_i	0	1	2	3
$f_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E[X] = \sum_x x_i\, f_X(x)$$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{0 + 3 + 6 + 3}{8} = \frac{12}{8} = 1.5$$

$$E[X^2] = \sum_x x_i^2\, f_X(x)$$

$$= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8}$$

$$= \frac{0 + 3 + 12 + 9}{8} = \frac{24}{8} = 3$$

$$E[X^3] = \sum_x x_i^3\, f_X(x)$$

$$= 0^3 \times \frac{1}{8} + 1^3 \times \frac{3}{8} + 2^3 \times \frac{3}{8} + 3^3 \times \frac{1}{8}$$

$$= \frac{0 + 3 + 24 + 81}{8} = \frac{108}{8} = 13.5$$

Variance: $\sigma_X^2 = m_2 - m_1^2 = 3 - 1.5^2 = 0.75$ (or) other method

$$\sigma_X^2 = E[(x - \overline{X})^2] = \sum_x = (x_i - \overline{X})^2\, f_X(x)$$

$$= \frac{1(0 - 1.5)^2 + 3(1 - 1.5)^2 + 3(2 - 1.5)^2 + 1(3 - 1.5)^2}{8}$$

$$= \frac{1.5^2 + 3(0.5^2) + 3(0.5^2) + 1(1.5^2)}{8} = \frac{2.25 + 0.75 + 0.75 + 2.25}{8} = \frac{6}{8}$$

$\therefore \sigma_X^2 = 0.75$

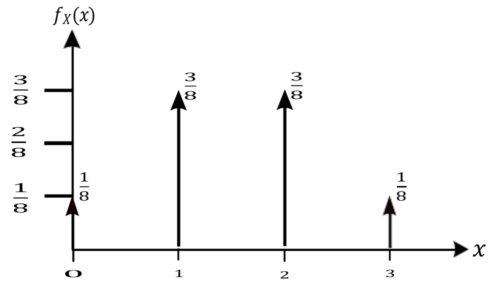
standard deviation: $\sigma_X = \sqrt{0.75} = 0.866$

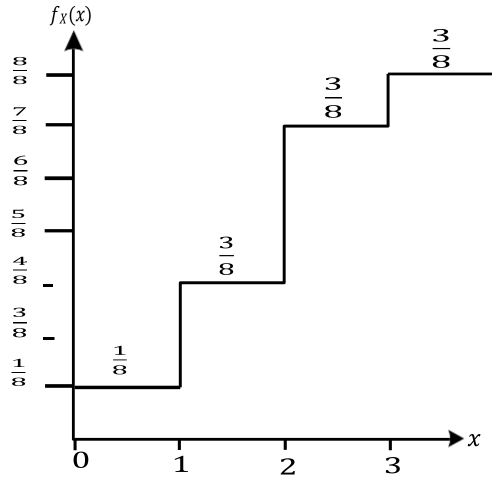
skew:

$\mu_3 = m_3 - 3m_1\sigma_X^2 - m_1^3 = 13.5 - 3 \times 1.5 \times 0.75 - 1.5^3$

$= 13.5 - 3.375 - 3.375 = 13.5$

Skewness: $\frac{\mu_3}{\sigma_X^3} = \frac{13.5}{0.866^3} = 20.7864$





CHAPTER 3

Binomial and Possion Random Variables

3.1 Binomial random variable

Let 'X' be the discrete r.v, the probability density function (PDF) can be written as

$$f_X(x) = P(X = x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k); \qquad k = 0, 1, 2, 3, \ldots N$$

where $\binom{N}{k} = \frac{N!}{(N-k)!k!}$

Here N no.of times random experiment is performed,

p, q are probabilities; $q = 1 - p$

The binamial distribution function (CDF) is

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} U(x-k); \qquad k = 0, 1, 2, 3, \ldots N$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

The special case of binomial distribution function with $N = 1$, then it is also called the Bernouli distribution.

Application:

1. The binomial density function is applied in the Bernouli trails experiment.
2. Bernouli experiment contains any two outcomes. For example
   - Hit or Miss of target in RADAR,
   - Pass or Fail in exam,
   - Hed or tail in a tossing a coin,
   - Winning or loosing of game,
   - Receiving '0' or '1'.

- the number of disk drives that crashed in a cluster of 1000 computers, and
- the number of advertisements that are clicked when 40,000 are served.

Problem: Let $N = 6, P = 0.25$ the find PDF and CDF for Binomial.

Solution: The PDF function is

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k); \qquad k = 0, 1, 2, 3, \ldots N$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$

$$f_X(x) = \sum_{k=0}^{6} \binom{6}{k} (0.25)^k (0.75)^{6-k} \delta(x-k);$$

$$= \binom{6}{0} (0.25)^0 (0.75)^6 \delta(x-0) + \binom{6}{1} (0.25)^1 (0.75)^5 \delta(x-1)$$

$$+ \binom{6}{2} (0.25)^2 (0.75)^4 \delta(x-2) + \binom{6}{3} (0.25)^3 (0.75)^3 \delta(x-3)$$

$$+ \binom{6}{4} (0.25)^4 (0.75)^2 \delta(x-4) + \binom{6}{5} (0.25)^5 (0.75)^1 \delta(x-5)$$

$$+ \binom{6}{6} (0.25)^6 (0.75)^0 \delta(x-6)$$

$$x = 0; f_X(0) = \binom{6}{0} (0.25)^0 (0.75)^6 \delta(0-0) = 0.1779$$

$$x = 1; f_X(1) = \binom{6}{1} (0.25)^1 (0.75)^5 \delta(1-1) = 0.35595$$

$$x = 2; f_X(2) = \binom{6}{2} (0.25)^2 (0.75)^4 \delta(2-2) = 0.29663$$

$$x = 3; f_X(3) = \binom{6}{3} (0.25)^3 (0.75)^3 \delta(3-3) = 0.13183$$

$$x = 4; f_X(4) = \binom{6}{4} (0.25)^4 (0.75)^2 \delta(4-4) = 0.03295$$

$$x = 5; f_X(5) = \binom{6}{5} (0.25)^5 (0.75)^1 \delta(5-5) = 0.00439$$

$$x = 6; f_X(6) = \binom{6}{6} (0.25)^6 (0.75)^0 \delta(6-6) = 0.000244$$

The CDF function is

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} U(x-k); \qquad k = 0, 1, 2, 3, \ldots N$$

$$F_X(0) = P(X \leq 0) = 0.1779$$

$$F_X(1) = P(X \leq 1) = P(X \leq 0) + P(X = 1) = 0.1779 + 0.3559 = 0.5339$$

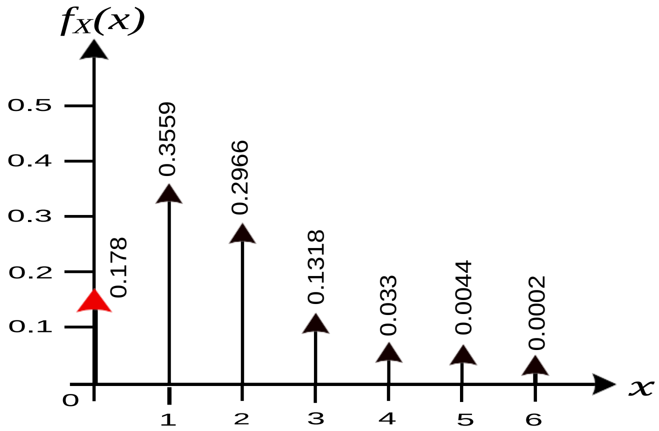
$$F_X(2) = P(X \le 2) = P(X \le 1) + P(X = 2) = 0.5339 + 0.2966 = 0.83043$$

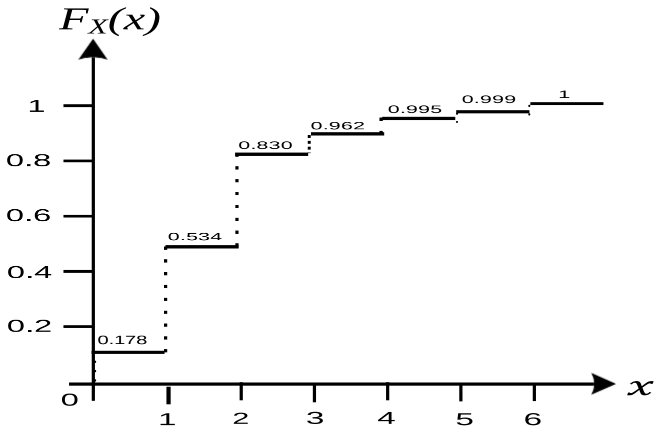
$$F_X(3) = P(X \le 3) = P(X \le 2) + P(X = 3) = 0.8343 + 0.1318 = 0.96223$$

$$F_X(4) = P(X \le 4) = P(X \le 3) + P(X = 4) = 0.9622 + 0.0329 = 0.99518$$

$$F_X(5) = P(X \le 5) = P(X \le 4) + P(X = 5) = 0.9951 + 0.0043 = 0.99957$$

$$F_X(5) = P(X \le 5) = P(X \le 5) + P(X = 6) = 0.9995 + 0.00024 = 0.9998$$





3.1.1 Statistical parameters of Binomial R.V

Let 'X' be the random variable, the PDF can be written as

$$f_X(x) = \binom{N}{x} p^x (1-p)^{N-x} = \binom{N}{x} p^x q^{N-x} = (p+q)^N$$

1. Mean Value

$$m_1 = E[X] = \sum x f_X(x)$$

$$= \sum_{x=0}^{N} x \cdot \binom{N}{x} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} x \cdot \frac{N!}{(N-x)!x!} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} x \cdot \frac{N(N-1)!}{(N-x)!\, x\,(x-1)!} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} \frac{N(N-1)!}{\left[(N-1)-(x-1)\right]!\,(x-1)!} p \cdot p^{x-1} q^{(N-1)-(x-1)}$$

$$= Np \sum_{x=0}^{N} \frac{(N-1)!}{\left[(N-1)-(x-1)\right]!\,(x-1)!} p^{x-1} q^{(N-1)-(x-1)}$$

$$= Np \binom{N-1}{x-1} p^{x-1} q^{(N-1)-(x-1)}$$

$$= Np \left[p+q\right]^{N-1} \qquad \because \binom{N}{x} p^x q^{N-x} = (p+q)^N$$

$$E[X] = Np \qquad \because p+q = 1$$

$$\therefore E[X] = Np$$

2. Mean Square Value

$$m_2 = E[X^2] = \sum_{x=0}^{N} x^2 f_X(x)$$

$$= \sum_{x=0}^{N} x^2 \cdot \binom{N}{x} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} \left[x(x-1) + x\right] \cdot \frac{N!}{(N-x)!\,x!} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} x(x-1) \cdot \frac{N!}{(N-x)!\,x!} p^x q^{N-x} + \underbrace{\sum_{x=0}^{N} x \cdot \frac{N!}{(N-x)!\,x!} p^x q^{N-x}}_{E[X]}$$

$$= \sum_{x=0}^{N} x(x-1) \cdot \frac{N!}{(N-x)!\,x(x-1)(x-2)!} p^x q^{N-x} + E[X]$$

$$= \sum_{x=0}^{N} \frac{N(N-1)(N-2)!}{\left[(N-2)-(x-2)\right]!\,(x-2)!} p^2 p^{x-2} q^{(N-2)-(x-2)} + E[X]$$

$$= N(N-1)p^2 \underbrace{\sum_{x=0}^{N} \frac{(N-2)!}{\left[(N-2)-(x-2)\right]!\,(x-2)!} p^{x-2} q^{(N-2)-(x-2)}}_{(p+q)^{N-2}} + \underbrace{E[X]}_{Np}$$

$$= N(N-1)\, p^2\, (p+q)^{N-2} + Np$$

$$= N^2 p^2 - N p^2 + Np \qquad \because p+q = 1$$

$$\therefore\ E[X^2] = N^2p^2 - Np^2 + Np$$

3. Third moment about origin

$$m_3 = E[X^3] = \sum_{x=0}^{N} x^3 f_X(x) \qquad \because\ x(x-)(x-2) = x^3 - 3x^2 + 2x$$

$$= \sum_{x=0}^{N} \left[x(x-1)(x-2) + 3x^2 - 2x \right] \cdot f_X(x)$$

$$= \sum_{x=0}^{N} x(x-1)(x-2) f_X(x) + 3\underbrace{\sum_{x=0}^{N} x^2 f_X(x)}_{E[X^2]} - 2\underbrace{\sum_{x=0}^{N} x f_X(x)}_{E[X]}$$

$$= \sum_{x=0}^{N} x(x-1)(x-2) f_X(x) + 3E[X^2] - 2E[X] \qquad \Rightarrow (1)$$

$$\text{Let} \sum_{x=0}^{N} x(x-1)(x-2) f_X(x)$$

$$= \text{Let} \sum_{x=0}^{N} x(x-1)(x-2) \binom{N}{x} p^x q^{N-x}$$

$$= \sum_{x=0}^{N} x(x-1)(x-2) \frac{N!}{(N-x)!x!} p^x q^{N-x}\, p^x q^{N-x}$$

$$= \sum_{x=0}^{N} x(x-1)(x-2) \frac{N(N-1)(N-2)(N-3)!}{x(x-1)(x-2)(x-3)!\ \left[(N-3)-(x-3)\right]} \times p^3 p^{x-3} q^{(N-3)-(x-3)}$$

$$= N(N-1)(N-2)\, p^3 \sum_{x=0}^{N} \frac{N(N-1)(N-2)(N-3)!}{(x-3)!\ (N-3)-(x-3)}\, p^{x-3} q^{(N-3)-(x-3)}$$

$$= N(N-1)(N-2)\, p^3 \sum_{x=0}^{N} \binom{N-3}{x-3} p^{x-3} q^{(N-3)-(x-3)}$$

$$= N(N-1)(N-2)\, p^3 (p+q)^{n-3}$$

$$= N(N-1)(N-2)\, p^3$$

$$= (N^3 - 2N^2 - N^2 + 2N)\, p^3$$

$$= N^3p^3 - 3N^2p^3 + 2Np^3$$

(1) $= N^3p^3 - 3N^2p^3 + 2Np^3 + 3\left(N^2p^2 - Np^2 + Np\right) - 2Np$

$$= N^3p^3 - 3N^2p^3 + 2Np^3 + 3N^2p^2 - 3Np^2 + 3Np - 2Np$$

$$= Np - 3Np^2 + 3N^2p^2 + 2Np^3 - 3N^2p^3 + N^3p^3$$

$$\therefore\ E[X^3] = Np - 3Np^2 + 3N^2p^2 + 2Np^3 - 3N^2p^3 + N^3p^3$$

4. Variance:

$$\mu_2 = \sigma_X^2 = E[X^2] - (E[X])^2$$
$$= N^2p^2 - Np^2 + Np - (Np)^2$$
$$= Np(1 - p) = Npq$$

Variance: $\sigma_X^2 = Npq$

5. Standard deviation : $\sigma_X = \sqrt{\mu_2} = \sqrt{Npq}$

6. Skew: μ_3

$$\mu_3 = E[X^3] - 3E[X]\sigma_X^2 - (E[x])^3$$
$$= Np - 3Np^2 + 3N^2p^2 + 2Np^3 - 3N^2p^3 + N^3p^3 - 3Np(Npq) - (Np)^3$$
$$= Np - 3Np^2 + 3N^2p^2 + 2Np^3 - 3N^2p^3 + N^3p^3 - 3N^2p^2(1 - p) - (Np)^3$$
$$= Np - 3Np^2 + \cancel{3N^2p^2} + 2Np^3 - \cancel{3N^2p^3} - \cancel{3N^2p^2} + \cancel{3N^2p^3}$$
$$= Np - 3Np^2 + 2Np^3$$
$$= Np(1 - 3p) + 2Np^3 = Np(q - 2p) + 2Np^3 \qquad \because q = 1 - p$$
$$= Npq - 2Np^2 + 2Np^3 = Npq - 2Np^2(1 - p)$$
$$= Npq - 2Np^2q = Npq(1 - 2p)$$

$$\therefore \mu_3 = Npq(1 - 2p)$$

7. Skewness:

$$\alpha = \frac{\mu_3}{\sigma_X^3} = \frac{Npq(1 - 2p)}{(Npq)^{\frac{3}{2}}} \qquad \because \sigma_X = (Npq)^{\frac{1}{2}}$$
$$= \frac{1 - 2p}{\sqrt{Npq}}$$

Problem: Find the probability in Tossing a fair coin five times, there will be appear

1. Getting three heads
2. Two heads and three tails
3. Atleast one head
4. not more than one tail

Solution:

Let p is the probability of getting head $p = 0.5$

q is the probability of getting tail $q = 1 - p = 0.5$

No.of times experiment performed $N = 5$

$$P[X = x] = f_X(x) = \binom{N}{x} p^x q^{N-x}$$

1.

P[Getting three heads] $= P[X = 3]$

$$= \binom{5}{3} 0.5^3 0.5^{5-3} = \frac{5!}{3!2!} 0.5^3 0.5^2$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} \times \frac{1}{32}$$

$$= \frac{10}{32} = 0.3125$$

2.

P[3 tails and 2 heads] $= \binom{5}{2} 0.5^2 0.5^3 = 0.3125$

3.

P[At least one head] $= P[X \geq 1]$

$$= \binom{5}{2} 0.5^2 0.5^3 = 0.3125$$

$$= 1 - \binom{5}{0} (0.5)^0 (0.5)^5$$

$$= 1 - \frac{1}{32} = 0.96875$$

4.

P[Not more than one tail] = Not more than one tail

$$= P[X = 0] + P[X = 1]$$

$$= \binom{5}{0} 0.5^0 0.5^5 + \binom{5}{1} 0.5^1 0.5^4$$

$$= 0.03125 + 0.15625 = 0.1875$$

Problem: If the mean and variance of binomial r.v 6 and 1.5 respectively. Find $E\left[X - P(X \geq 3)\right]$ and also find its PDF and CDF.

Solution: Given Mean value $E[X] = Np = 6 \longrightarrow (1)$

Variance $\sigma_X^2 = Npq = 1.5 \longrightarrow (2)$

$(2) \Rightarrow 6q = 1.5 \quad \because (1)$

$\Rightarrow q = 0.25$

$p = 1 - q = 0.75$

$(1) \Rightarrow N(0.75) = 6 \Rightarrow N = 8$

$\therefore p = 0.75;$ $\quad q = 0.25;$ $\quad N = 8$

$E\left[X - P(X \geq 3)\right] = E[X] - E[P(X \geq 3)] \quad \Longrightarrow (3)$

$$E[P(X \geq 3)] = 1 - P[X < 3]$$

$$= 1 - \{P[X = 0] + P[X = 0] + P[X = 0]\}$$

$$P[X = 0] = f_X(0) = \binom{8}{0} 0.75^0 0.25^8 = 1 \times 1 \times 0.25^8 = 1.525 \times 10^{-5}$$

$$P[X = 1] = f_X(1) = \binom{8}{1} 0.75^1 0.25^7 = 8 \times 0.75 \times 0.25^7 = 3.662 \times 10^{-4}$$

$$P[X = 2] = f_X(2) = \binom{8}{2} 0.75^2 0.25^6 = 1 \times 0.75^2 \times 0.25^6 = 3.8452 \times 10^{-3}$$

$$\therefore E[P(X \geq 3)] = 1 - P[X < 3]$$

$$= 1 - 4.2239 \times 10^{-3}$$

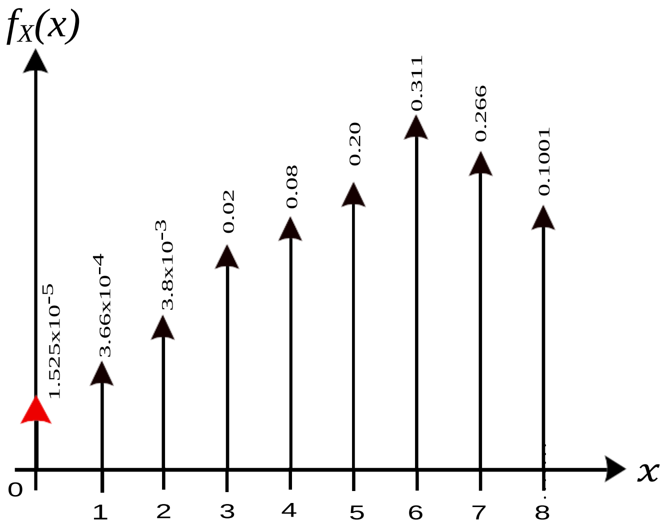
$$= 0.995$$

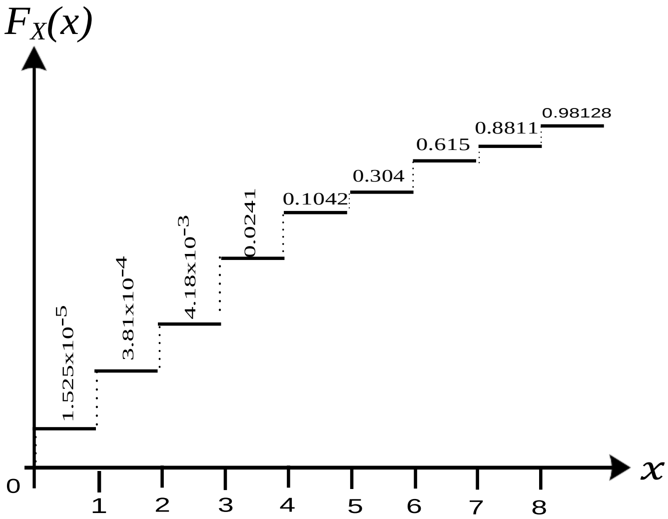
$$\Longrightarrow ③$$

$$E\left[X - P(X \geq 3)\right] = E[X] - E[P(X \geq 3)] = 6 - 0.995 = 5.005$$

$$\boxed{\therefore E\left[X - P(X \geq 3)\right] = 5.005}$$

PDF and CDF:





$$P(X = 3) = f_X(3) = \binom{8}{3} 0.75^3 0.25^5 = 0.0230$$

$$P(X = 4) = f_X(4) = \binom{8}{4} 0.75^4 0.25^4 = 0.0865$$

$$P(X = 5) = f_X(5) = \binom{8}{5} 0.75^5 0.25^3 = 0.20764$$

$$P(X = 6) = f_X(6) = \binom{8}{6} 0.75^6 00.25^2 = 0.34446$$

$$P(X = 7) = f_X(7) = \binom{8}{7} 0.75^7 0.25^1 = 0.26696$$

$$P(X = 8) = f_X(8) = \binom{8}{8} 0.75^8 0.25^0 = 0.10011$$

3.2 Possion random variable

Let 'X' be the random variable with probability density function

$$f_X(x) = \frac{e^{-b}\, b^x}{x!}; \quad x = 0, 1, 2, \ldots \infty$$

Where $b = \lambda T$ and $b > 0$, real constant

T = Time duration in which number of events are conducted.

λ = The average number of events per time

Let constant $b = 2$, plot PDF and CDF.

We know that $f_X(x) = P[X = x] = \dfrac{e^{-b}\, b^x}{x!}$

Given that $b = 2$

at $x = 0 \Rightarrow P[X = 0] = f_X(0) = \dfrac{e^{-2}\, 2^0}{0!} = 0.135$

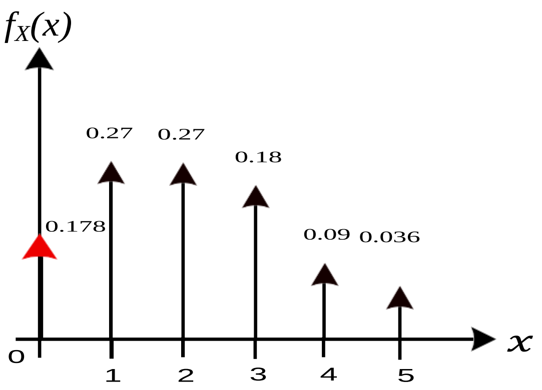
at $x = 1 \Rightarrow P[X = 1] = f_X(1) = \dfrac{e^{-2}\, 2^1}{1!} = 0.27$

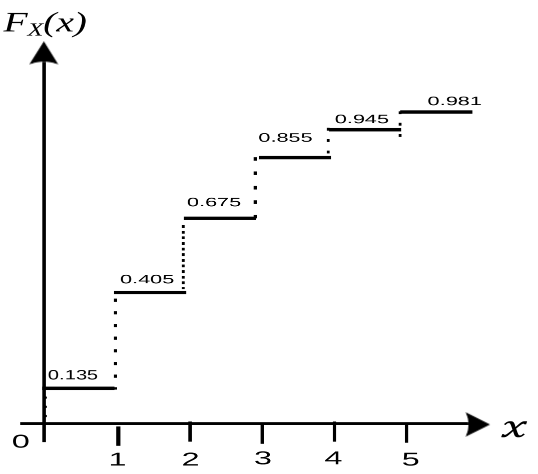
at $x = 2 \Rightarrow P[X = 2] = f_X(2) = \dfrac{e^{-2}\, 2^2}{2!} = 0.27$

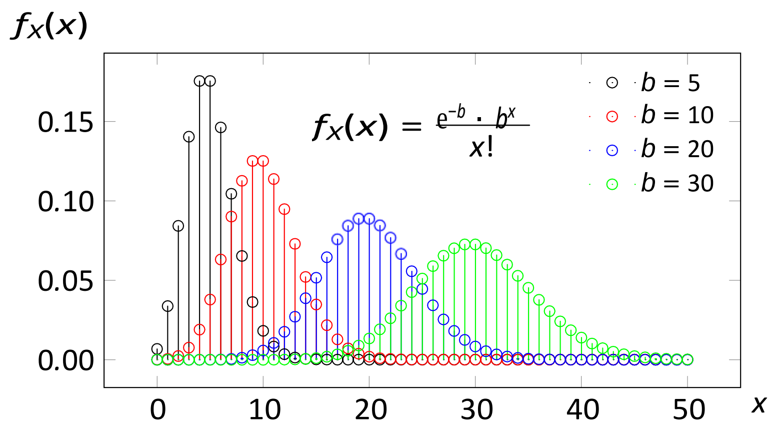
at $x = 3 \Rightarrow P[X = 3] = f_X(3) = \dfrac{e^{-2}\, 2^3}{3!} = 0.18$

at $x = 4 \Rightarrow P[X = 4] = f_X(4) = \dfrac{e^{-2}\, 2^4}{4!} = 0.09$

at $x = 5 \Rightarrow P[X = 5] = f_X(5) = \dfrac{e^{-2}\, 2^5}{5!} = 0.036$







Applications:

1. It is used in counting applications.

    - Number of vehicles arrived at a petrol pump
    - Number of customers arrived at super market
    - Number of account holders arrived at bank

2. It is describes

    - The number of units in a sample taken from a production line.
    - The number of telephone calls made during a period of time.
    - The number of electrons emmitted from a small section of a cathode in a given time.

3.2.1 Statistical parameter of Possion random variable

Let 'X' be the random variable with probability density function defined as

$$f_X(x) = \frac{e^{-b}\, b^x}{x!}; \quad x = 0, 1, 2, \ldots$$

1. Mean value

$$E[X] = \overline{X} = m_1 = \sum_{x=0}^{\infty} x f_X(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-b}\, b^x}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} \cancel{x} \frac{b \cdot b^{x-1}}{\cancel{x}(x-1)!}$$

$$= b\, e^{-b} \sum_{x=1}^{\infty} \frac{b^{x-1}}{(x-1)!}$$

$$= b\, e^{-b}\left(1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots\right)$$

$$= b\, e^{-b} \times e^{b} = b \qquad \because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\boxed{\therefore\ E[X] = m_1 = b}$$

2. Mean squre value

$$E[X^2] = \overline{X^2} = m_2 = \sum_{x=0}^{\infty} x^2 f_X(x)$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-b}\, b^x}{x!}$$

$$= \sum_{x=0}^{\infty} \left[x(x-1) + x\right] \frac{e^{-b}\, b^x}{x!} \qquad \because x(x-1) + x = x^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-b}\, b^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-b}\, b^x}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x(x-1)} \frac{e^{-b}\, b^2\, b^{x-2}}{\cancel{x(x-1)}\,(x-2)!} + E[X]$$

$$= e^{-b}\, b^2 \sum_{x=2}^{\infty} \frac{b^{x-2}}{(x-2)!} + b \qquad \because E[X] = b$$

$$= e^{-b}\, b^2 \left(1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots\right) + b$$

$$= \cancel{e^{-b}}\, b^2 \times \cancel{e^{b}} + b = b + b^2 \qquad \because \boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots}$$

$$\boxed{\therefore\ E[X^2] = m_2 = b + b^2}$$

3. Third moment about origin

$$E[X^3] = \overline{X^3} = m_3 = \sum_{x=0}^{\infty} x^3 f_X(x)$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-b}\, b^x}{x!} \qquad \because x(x-1)(x-2) + 3x^2 - 2x = x^3$$

$$= \sum_{x=0}^{\infty} \left[x(x-1)(x-2) + 3x^2 - 2x\right] \frac{e^{-b}\, b^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-b}\, b^x}{x!} + \sum_{x=0}^{\infty} 3x^2 \frac{e^{-b}\, b^x}{x!} + \sum_{x=0}^{\infty} (-2x) \frac{e^{-b}\, b^x}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x(x-1)(x-2)} \frac{e^{-b}\, b^3\, b^{x-3}}{\cancel{x(x-1)(x-2)}(x-3)!} + 3E[X^2] - 2E[X]$$

$$= e^{-b} \sum_{x=2}^{\infty} \frac{b^{x-2}}{(x-2)!} + 3[b + b^2] - 2[b] \quad \because \boxed{E[X^2] = b + b^2;\ E[X] = b}$$

$$= e^{-b} b^3 \left[1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots\right] + 3b^2 + 3b - 2b$$

$$= e^{-b} b^3 \times e^{b} + 3b^2 + b = b^3 + 3b^2 + b \quad \because \boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots}$$

$$\boxed{\therefore\ E[X^3] = m_3 = b^3 + 3b^2 + b}$$

4. Variance: $\sigma_X^2 = m_2 - m_1^2 = b + b^2 - (b)^2 = b$

5. Standard deviation: $\sigma_X = \sqrt{b}$

6 . Skew:

$$\mu_3 = E[X^3] - 3E[X]\sigma^2_X - (E[X])^3$$

$$= b^3 + 3b^2 + b - 3b(b) - (b)^3$$

$$= b$$

7. Skewness: $\alpha = \frac{\mu_3}{\sigma_X^3} = \frac{b}{b^{\frac{3}{2}}} = \frac{1}{\sqrt{b}}$

Problem: Assume vehicles arrived at petrol bunk follows possion random variable and occured at average rate of 50 per hour. The petrol bunk has only one station. It is assumed that one munute is required to obtain fuel. What is the probability that a waiting line will occur at the petrol bunk?

Solution: Given

Average rate of arriaval of cars $\lambda = 50/hour = \frac{50}{60} = 0.833$

Time required to filling fuel $T = 1\ min$

$$b = \lambda t = 0.833 \times 1 = 0.833$$

A waiting line willoccur if two or more vehicles in a minute.

Probability of a waiting line = $P(waiting) = 1 - [P[x = 0] + P[X = 1]]$

$$P[X = x] = f_X(x) = \frac{e^{-b}\, b^x}{x!}$$

$$P[X = 0] = f_X(0) = \frac{e^{-0.833}\ 0.833^0}{0!} = 0.4347$$

$$P[X = 1] = f_X(1) = \frac{e^{-0.833}\ 0.833^1}{1!} = 0.3621$$

$\therefore\ P(waiting) = 1 - (0.4347 + 0.3621) = 0.2032$

Problem: A random variable is known to be possion with constant $b = 4$. Find the probability of event $P\{0 \le X \le 5\}$.

Solution: Given $b = 4$

$$P\{0 \le X \le 5\} = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3]$$
$$+ P[X = 4] + P[X = 5]$$
$$= 0.0183 + 0.0732 + 0.1464 + 0.1952 + 0.1952 + 0.156$$
$$= 0.7843$$

CHAPTER 4

Probability Generating Function

4.1 Functions that give moments

Two functions can be defined that allow moments to be calculated for random variable 'X'. They are

1. Characteristic function 2. Moment Generating function

4.1.1 Characteristic function

The characteristic function of random variable X is dfined by

$$\Phi_X(\omega) = E[e^{j\omega X}]; \qquad j = \sqrt{-1}; \text{ and } -\infty \leq \omega \leq \infty$$

$$= \int_{x=-\infty}^{\infty} f_X(x)e^{j\omega x} \, d\omega$$

where $f_X(x)$ is probability density function

It is similar to Fourier Transform with sign reversed inh exponential. Therefore, the PDF function defined as

$$f_X(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \Phi_X(\omega) \, e^{-j\omega x} \, d\omega$$

The PDF and characteristic function are Fourier Tranform pair.

$$f_X(x) \underset{\text{IFT}}{\overset{\text{FT}}{\Longleftrightarrow}} \Phi_X(\omega)$$

n^{th} moment can be obtained by derivating $\Phi_X(\omega)$ in 'n' times with respect to ω and setting $\omega = 0$.

$$m_n = (-j)^n \left. \frac{d^n\Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}; \qquad n = 1, 2, 3, \ldots$$

$$m_1 = (-j)^1 \left. \frac{d\Phi_X(\omega)}{d\omega} \right|_{\omega=0}$$

$$m_2 = (-j)^2 \left. \frac{d^2\Phi_X(\omega)}{d\omega^2} \right|_{\omega=0}$$

$$m_3 = (-j)^3 \left. \frac{d^3\Phi_X(\omega)}{d\omega^3} \right|_{\omega=0}$$

Problem 1: Find the moment of exponential PDF of continuous r.v 'X' is given

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}}; & x \geq a \\ 0; & x < a \end{cases}$$

using characteristics function

Solution: Characteristic function

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, d\omega$$

$$= \int_{x=a}^{\infty} \frac{1}{b} e^{-\frac{(x-a)}{b}}\, e^{j\omega x}\, d\omega$$

$$= \frac{1}{b} e^{\frac{a}{b}} \int_{x=a}^{\infty} e^{-\frac{x}{b}}\, e^{j\omega x}\, d\omega$$

$$= \frac{e^{\frac{a}{b}}}{b} \int_{x=a}^{\infty} e^{-\left(\frac{1}{b} - j\omega\right)x}\, d\omega$$

$$= \frac{e^{\frac{a}{b}}}{b} \left[\frac{e^{-\left(\frac{1}{b} - j\omega\right)x}}{-\left(\frac{1}{b} - j\omega\right)} \right]_{x=a}^{\infty}$$

$$= \frac{e^{\frac{a}{b}}}{b} \left[\frac{e^{-\infty} - e^{-\left(\frac{1}{b} - j\omega\right)a}}{-\left(\frac{1}{b} - j\omega\right)} \right]$$

$$= \frac{e^{\frac{a}{b}}}{\not{b}} \left[\frac{0 - e^{-\frac{a}{b}} \cdot e^{+j\omega a}}{-\frac{1-j\omega b}{\not{b}}} \right]$$

$$= e^{\frac{a}{b}} \left[\frac{e^{-\frac{a}{b}} \cdot e^{+j\omega a}}{1 - j\omega b} \right]$$

$$\Phi_X(\omega) = \frac{e^{j\omega a}}{1 - j\omega b}$$

$$\boxed{\therefore\ \Phi_X(\omega) = \frac{e^{j\omega a}}{1 - j\omega b}}$$

First moment $m_1 = (-j)^1 \left. \frac{d\Phi_X(\omega)}{d\omega} \right|_{\omega=0} \longrightarrow (1)$

$$\frac{d\Phi_X(\omega)}{d\omega} = \frac{d}{d\omega}\left[\frac{e^{j\omega a}}{1-j\omega b}\right]_{\omega=0}$$

$$= \left.\frac{(1-j\omega b)\frac{d}{d\omega}\left(e^{j\omega a}\right) - e^{j\omega a}\frac{d}{d\omega}(1-j\omega b)}{(1-j\omega b)^2}\right|_{\omega=0}$$

$$= \left.\frac{(1-j\omega b)\left(e^{j\omega a}\cdot ja\right) - e^{j\omega a}\left(0-jb\right)}{(1-j\omega b)^2}\right|_{\omega=0}$$

$$= ja + jb$$

$$\therefore \text{(1)} \Rightarrow m_1 = (-j)\left[ja+jb\right] = -j^2(a+b) = a+b$$

$$\boxed{\therefore m_1 = a+b}$$

Problem 2: Find the density function whose characteristic function is

$$\Phi_X(\omega) = e^{|\omega|}; \quad -\infty < \omega < \infty$$

Solution: Given

$$\Phi_X(\omega) = \begin{cases} e^{\omega} & \omega < 0 \\ e^{-\omega} & \omega \geq 0 \end{cases}$$

$$f_X(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \Phi_X(\omega)e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{\omega=-\infty}^{0} e^{j\omega}\cdot e^{-j\omega x}\, d\omega + \int_{\omega=0}^{\infty} e^{-j\omega}\cdot e^{-j\omega x}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{\omega=-\infty}^{0} e^{(1-jx)\omega}\, d\omega + \int_{\omega=0}^{\infty} e^{-(1+jx)\omega}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left.\frac{e^{(1-jx)\omega}}{1-ix}\right|_{-\infty}^{0} + \left.\frac{e^{(1-jx)\omega}}{1-ix}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} - 0 + 0 - \left[-\frac{1}{1+jx}\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} + \frac{1}{1+jx}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1+jx+1-jx}{1^2-(jx)^2}\right] = \frac{1}{2\pi}\left[\frac{2}{1+x^2}\right]$$

$$f_X(x) = \frac{1}{\pi}\,\frac{1}{1+x^2}$$

Problem 3: A random variable 'X' has a characteristic function given by

$$\Phi_X(\omega) = \begin{cases} 1 - |\omega|; & |\omega| \le 1 \\ 0; & \text{other wise} \end{cases}$$

Solution:

$$f_X(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \Phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-1}^{1} \Phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-1}^{0} (1+\omega)\, e^{-j\omega x}\, d\omega + \frac{1}{2\pi}\int_{\omega=0}^{1} (1-\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{(j\omega)^2}\left(e^{jx} - 1\right) + \frac{1}{(j\omega)^2}\left(e^{-jx} - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(j\omega)^2}\left(e^{jx} - + e^{-jx} - 1\right)\right]$$

$$= \frac{1}{\pi x^2}\left[1 - \frac{e^{jx} + e^{-jx}}{2}\right]$$

$$= \frac{1}{\pi x^2}\left[1 - \cos x\right]$$

$$f_X(x) = \frac{1 - \cos x}{\pi}$$

Problem 4: A characteristic function of a r.v is given by $\Phi_X(\omega) = \frac{1}{(1-j2\omega)^{\frac{N}{2}}}$ Find the mean, mean square and variance of a r.v X.

Solution: n^{th} moment

$$m_n = (-j)^n \frac{d^n}{d\omega^n}\Phi_X(\omega)\Big|_{\omega=0}$$

1. First moment

$$m_1 = (-j)\frac{d}{d\omega}\left[\frac{1}{(1-j2\omega)^{\frac{N}{2}}}\right]_{\omega=0}$$

$$= (-j)\frac{d}{d\omega}\left[(1-j2\omega)^{-\frac{N}{2}}\right]_{\omega=0}$$

$$= (-j)\left[-\frac{N}{2}(1-j2\omega)^{-\frac{N}{2}-1}\frac{d}{d\omega}(1-j2\omega)\right]_{\omega=0}$$

$$= -j\left(-\frac{N}{2}\right)(1-j2\omega)^{-\frac{N}{2}-1}\cdot(0-j2).$$

$$= -j^2\,\frac{2N}{2}\,(1-j2\omega)^{-\frac{N}{2}-1}\Big|_{\omega=0}$$

$$= +N(1-0)^{-\frac{N}{2}-1} = N$$

$$\boxed{\therefore\ m_1 = N}$$

2. Second moment

$$m_2 = (-j)^2\,\frac{d^2}{d\omega^2}\left(1-j2\omega\right)^{-\frac{N}{2}}\Big|_{\omega=0}$$

$$= (-1)\frac{d}{d\omega}\left[-\frac{N}{2}\,(1-j2\omega)^{-\frac{N}{2}-1}\times(-j2)\right]\Big|_{\omega=0}$$

$$= (-jN)\frac{d}{d\omega}\left(1-j2\omega\right)^{-\frac{N}{2}-1}\Big|_{\omega=0}$$

$$= -jN\left[\left(-\frac{N}{2}-1\right)(1-j2\omega)^{-\frac{N}{2}-2}\times(-j2)\right]\Big|_{\omega=0}$$

$$= +j^2\,2N\left[\frac{-N-2}{2}\,(1-j2\omega)^{-\frac{N}{2}-2}\right]\Big|_{\omega=0}$$

$$= -j^2N\left[(N+2)(1-j2\omega)^{-\frac{N}{2}-2}\right]\Big|_{\omega=0}$$

$$m_2 = N(N+2)$$

3. Variance: $\sigma_X^2 = m_2 - m_1^2 = N^2 + 2N - N^2 = 2N$

Problem 5: A r.v 'X' is defined by density function

$$f_X(x) = \begin{cases} 0; & x < 0 \\ 1; & 0 < x \\ 0; & x > 1 \end{cases}$$

$$\Phi_X(\omega) = \int_{x=-\infty}^{\infty} f_X(x)e^{j\omega x}\,dx$$

$$= \int_{x=0}^{1}(1)e^{j\omega x}dx = \left[\frac{e^{j\omega x}}{j\omega}\right]_0^1 = \frac{e^{j\omega}}{j\omega} - \frac{1}{j\omega} = \frac{e^{j\omega}-1}{j\omega}$$

4.1.2 Properties of characteristic function

1. The maximum value of characteristic function is unity

   Proof.

   $$\Phi_X(\omega) = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

   The max value $= |\Phi_X(\omega)|$

   $$= \left| \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \right|$$

   $$= \left| \int_{x=-\infty}^{\infty} f_X(x)\, \underbrace{\left| e^{j\omega x} \right|}_{=1}\, dx \right| \qquad \because\ |e^{i\vartheta}| = |\cos\vartheta + i\sin\vartheta|$$

   $$= \left| \int_{x=-\infty}^{\infty} f_X(x)\, dx \right| \qquad = \sqrt{\cos^2\vartheta + \sin^2\vartheta} = 1$$

   $$= 1$$

   □

2. If $f_X(x)$ is symmetric function then $\Phi_X(\omega)$ is also symmetric function.

   Proof.

   $$\Phi_X(\omega) = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

   let $x = -y \Rightarrow dx = -dy$

   If $x = -\infty \Rightarrow y = +\infty$ and If $x = \infty \Rightarrow y = -\infty$

   $$\Phi_X(\omega) = \int_{y=\infty}^{-\infty} f_X(-y)\, e^{j\omega(-y)}\, (-dy)$$

   $$= \int_{y=-\infty}^{\infty} f_X(y)\, e^{j\omega(-y)}\, (dy)$$

   $$\Phi_X(\omega) = \Phi_X(-\omega)$$

   □

3. If X and Y are sum of two independent random variable, then the characteristic function is $\Phi_{X+Y}(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$

Proof.

$$\Phi_X(\omega) = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$\Phi_{X+Y}(\omega) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{XY}(x, y)\, e^{j\omega(x+y)}\, dy\, dx$$

Independent r.v $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

$$= \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \cdot + \int_{x=-\infty}^{\infty} f_Y(y)\, e^{j\omega y}\, dy$$

$$\Phi_{X+Y}(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

□

4. If $\Phi_X(\omega)$ is characteristic function of X then $\Phi_{aX+b}(\omega) = e^{j\omega b}\, \Phi_X(a\omega)$

Proof.

$$\Phi_X(\omega) = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$\Phi_{aX+b}(\omega) = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega(ax+b)}\, dx$$

$$= e^{j\omega b}\int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega(ax)}\, dx$$

$$= e^{j\omega b}\, \Phi_X(a\omega)$$

□

Problem: Find the characteristic function of following PDF. (a) Uniform (b) Exponential (c) Gaussian (d) Rayleigh (e) Binomial (f) Possion

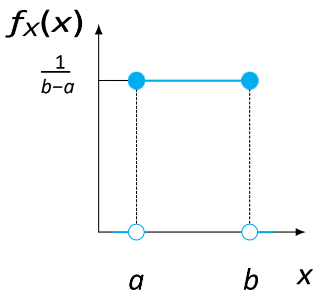
Solution:

(a) Uniform r.v

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_a^b e^{j\omega x} f_X(x)\, dx$$

$$= \frac{1}{b-a}\int_a^b e^{j\omega x}\, dx = \frac{1}{b-a}\left[\frac{e^{j\omega x}}{j\omega}\right]_a^b$$

$$\Phi_X(\omega) = \frac{e^{jb\omega} - e^{ja\omega}}{(b-a)\, j\omega}$$



$j\omega(b-a)$

(b) Exponential r.v

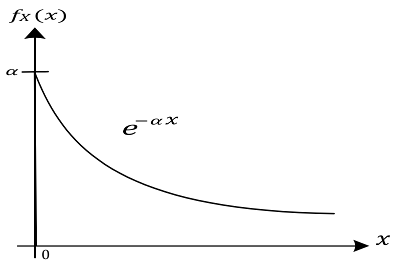
$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=0}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{j\omega x} \cdot \alpha e^{-\alpha x}\, dx$$

$$= \alpha \int_0^{\infty} e^{x(j\omega-\alpha)}\, dx = \alpha \left[\frac{e^{x(j\omega-\alpha)}}{j\omega-\alpha} \right]_0^{\infty}$$

$$= \alpha \left[\frac{e^{-x(\alpha-j\omega)}}{j\omega-\alpha} \right]_0^{\infty} = \alpha \left[0 - \frac{1}{j\omega-\alpha} \right]$$

$$= -\frac{\alpha}{j\omega-\alpha} = \frac{\alpha}{\alpha-j\omega}$$



(c) Gaussian r.v

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx$$

let $m = \frac{x-m}{\sigma} = t \Rightarrow dx = \sigma\, dt$

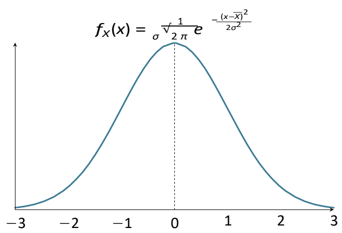
If $x = \pm\infty \Rightarrow t = \pm\infty$

$$= \frac{1}{\not\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega(m+\sigma t)}\, e^{-\frac{t^2}{2}} \not\sigma\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega m} \cdot e^{j\omega\sigma t} \cdot e^{-\frac{t^2}{2}}\, dt$$

$$= \frac{e^{j\omega m}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(t-\sigma j\omega)^2}{2}} \cdot e^{-\frac{\sigma^2\omega^2}{2}}\, dt$$

$$= \frac{e^{j\omega m - \frac{\sigma^2\omega^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(t-j\sigma\omega)^2}{2}}\, dt$$



let $y = t - j\sigma\omega \Rightarrow dt = dy$

$$= \frac{e^{j\omega m - \frac{\sigma^2\omega^2}{2}}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy}_{\text{even function}}$$

$$= \frac{e^{j\omega m - \frac{\sigma^2\omega^2}{2}}}{\sqrt{2\pi}}\, 2 \cdot \int_0^\infty e^{-\frac{y^2}{2}}\, dy$$

$$= \frac{e^{j\omega m - \frac{\sigma^2\omega^2}{2}}}{\sqrt{2\pi}} \times 2 \times \sqrt{\frac{\pi}{2}}$$

$$\Phi_X(\omega) = e^{j\omega m - \frac{\sigma^2\omega^2}{2}}$$

$$\boxed{\therefore\ \Phi_X(\omega) = e^{j\omega m - \frac{\sigma^2\omega^2}{2}}}$$

(d) Rayleigh distribution

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$= \int_0^\infty \frac{x}{\alpha^2}\, e^{\frac{-x^2}{2\alpha^2}} \cdot e^{j\omega x}\, dx$$

let $\frac{x}{\alpha} = t \Rightarrow dx = dt$

If $x = 0 \Rightarrow t = 0$; If $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\cancel{\alpha^2}} \int_0^\infty \cancel{\alpha}\, t\, e^{-\frac{t^2}{2}}\, e^{j\omega(\alpha t)}\, \cancel{\alpha}\, dt$$

$$= \int_0^\infty t\, e^{-\frac{t^2}{2}}\, e^{j\omega\alpha t}\, dt$$

$$= \int_0^\infty t\, e^{-\frac{t^2}{2} + j\omega\alpha t}\, dt$$

$$= \left[t \cdot \frac{e^{-\frac{t^2}{2} + j\omega\alpha t}}{-t + j\omega\alpha} \right]_0^\infty - \int_0^\infty \frac{e^{-\frac{t^2}{2} + j\omega\alpha t}}{-t + j\omega\alpha}\, dt$$

$$= \left[0 - 0 \right] - \left[\frac{e^{-\frac{t^2}{2} + j\omega\alpha t}}{(-t + j\omega\alpha)^2} \right]_{t=0}^\infty$$

$$= -\left[0 - \frac{1}{(j\omega\alpha)^2} \right] = \frac{1}{(j\omega\alpha)^2}$$

$$= -\frac{1}{\omega^2\alpha^2}$$

$$\boxed{\therefore\ \Phi_X(\omega) = -\frac{1}{\omega^2\alpha^2}}$$

(e) Binomial Distribution

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$= \sum_{x=0}^{N} \binom{N}{x} p^x q^{N-x}\, e^{j\omega x}$$

$$= \sum_{x=0}^{N} \binom{N}{x} (p\, e^{j\omega})^x q^{N-x} \quad \longrightarrow ①$$

$$= (p\, e^{j\omega} + q)^N$$

To prove the above statement, take $n = 2$;

$$① \Rightarrow \binom{2}{0} (p\, e^{j\omega})^0 q^2 + \binom{2}{x} (p\, e^{j\omega})^1 q^1 + \binom{2}{x} (p\, e^{j\omega})^2 q^2$$

$$\Rightarrow q^2 + 2(pe^{j\omega})q + (pe^{j\omega})^2$$

$$= (p\ e^{j\omega+q})^2$$ Hence proved

$$\therefore \Phi_X(\omega) = \sum_{x=0}^{N} \binom{N}{x} (p\, e^{j\omega})^x q^{N-x} = (p\, e^{j\omega} + q)^N$$

(f) Possion distribution

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$= \sum_{x=0}^{\infty} \frac{b^x\, e^{-b}}{x!}\, e^{j\omega x}\, dx$$

$$= e^{-b} \left[1 + \frac{be^{j\omega}}{1!} + \frac{(be^{j\omega})^2}{2!} + \ldots \right]$$

$$= e^{-b} \left[e^{be^{j\omega}} \right]$$

$$= e^{-(1-e^{j\omega})}$$

$$\therefore \Phi_X(\omega) = e^{-(1-e^{j\omega})}$$

4.1.3 Moment Generating Function (MGF):

Let 'X' be the random variable and PDF is $f_X(x)$ and the moment generating function is defined as

$$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{vx}\, dx$$

Apply Taylor series for e^{vX}

$$e^{vX} = 1 + vX + \frac{v^2X^2}{2!} + \frac{v^3X^3}{3!} + \ldots + \frac{v^rX^r}{r!} + \ldots + \frac{v^nX^n}{n!} + \ldots$$

Apply expectation both sides, then

$$E[e^{vX}] = 1 + vE[X] + \frac{v^2}{2!}E[X^2] + \frac{v^3}{3!}E[X^3] + \ldots + \frac{v^r}{r!}E[X^r] + \ldots + \frac{v^n}{n!}E[X^n] + \ldots$$

$$\therefore\ M_X(v) = E[e^{vX}] = 1 + vE[X] + \frac{v^2}{2!}E[X^2] + \frac{v^3}{3!}E[X^3] + \ldots + \frac{v^n}{n!}E[x^n] + \ldots$$

Differentiate the above equation with respect to 't' and then putting $t = 0$, we get

$$\frac{d}{dv}M_X(v)\Big|_{v=0} = m_1 \qquad \frac{d^2}{dv^2}M_X(v)\Big|_{v=0} = m_2$$

$$\boxed{\therefore\ m_n = \frac{d^n}{dv^n}M_X(v)\Big|_{v=0}; \quad n = 1, 2, 3, \ldots}$$

Problem: Prove that the MGF of random variable 'X' having PDF

$$f_X(x) = \begin{cases} \frac{1}{3}; & -1 < x < 2 \\ 0; & \text{otherwise} \end{cases}$$

is given by

$$M_X(v) = \begin{cases} \frac{e^{2v} - e^{-v}}{3v}; & v \quad 0 \\ 1; & v = 1 \end{cases}$$

Solution:

$$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x) e^{vx}\, dx$$

$$= \int_{x=-1}^{2} \frac{1}{3} e^{vx}\, dx$$

$$= \frac{1}{3}\left[\frac{e^{vx}}{v}\right]_{x=-1}^{2}$$

$$= \frac{1}{3}\left[\frac{e^{2v}}{v} - \frac{e^{-v}}{v}\right]$$

$$= \frac{e^{2v} - e^{-v}}{3v}$$

If $v = 0$; then $M_X(v) = \frac{0}{0}$; so, differentiate the $M_X(v)\big|_{v=0}$

$$= \left.\frac{\frac{e^{2v}(2) - e^{-v}(-1)}{3(1)}}{}\right|_{v=0}$$

$$= \frac{2+1}{3} = 1 \qquad \text{Hence proved}$$

Problem: Find the MGF, mean, mean squre and variance for given function of unifom random variable 'X'.

(a) $f_X(x) = \begin{cases} \frac{1}{b-a}; & a \le x \le b \\ 0; & \text{otherwise} \end{cases}$ (b) $f_X(x) = \begin{cases} \frac{1}{2a}; & -a \le x \le a \\ 0; & \text{otherwise} \end{cases}$

(c) $f_X(x) = \frac{1}{b} e^{-\frac{x}{b}}$

Solution:

(a) $M_X(v) = \int_a^b f_X(x) e^{vx}\, dx$

$$= \int_a^b \frac{1}{b-a} e^{vx}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{vx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{vx}}{v}\right]_a^b$$

$$= \frac{1}{b-a} \frac{e^{bv} - e^{av}}{v}$$

(b) $M_X(v) = \int_{-a}^a f_X(x) e^{vx}\, dx$

$$= \int_{-a}^a \frac{1}{2a} e^{vx}\, dx$$

$$= \frac{1}{2a} \int_{-a}^a e^{vx}\, dx$$

$$= \frac{1}{2a} \left[\frac{e^{vx}}{v}\right]_{-a}^a$$

$$= \frac{1}{2a} \left[\frac{e^{-av} - e^{av}}{v}\right]$$

$$= \frac{1}{av} \frac{e^{-av} - e^{av}}{2}$$

$$= \frac{\sin ax}{av}$$

Problem: Find moment generating function of a r.v 'X' having PDF

$$f_X(x) = \begin{cases} x; & 0 \le x \le 1 \\ 2 - x; & 1 \le x \le 2 \\ 0; & \text{elsewhere} \end{cases}$$

Solution:

$$M_X(v) = \int_{-a}^a f_X(x) e^{vx}\, dx$$

$$= \int_{x=0}^{1} x\, e^{vx}\, dx + \int_{x=1}^{2} (2-x) e^{vx}\, dx$$

$$= \left[x \cdot \frac{e^{vx}}{v} - \frac{e^{vx}}{v^2} \right]_0^1 + \left[(2-x) \frac{e^{vx}}{v} - (-1) \frac{e^{vx}}{v^2} \right]_1^2$$

$$= \left[\frac{e^v}{v} - \frac{e^{vx}}{v^2} + \frac{1}{v^2} \right] + \left[\frac{e^{2v}}{v^2} - \frac{e^v}{v} - \frac{e^v}{v^2} \right]$$

$$= \left[-\frac{e^v}{v^2} + \frac{1}{v^2} + \frac{e^{2v}}{v^2} - \frac{e^v}{v^2} \right]$$

$$= \frac{1}{v^2} \left[1 - 2e^v + e^{2v} \right]$$

$$M_X(v) = \left[\frac{1 - e^v}{v} \right]^2$$

Problem: The MGF of a r.v 'X' is given by $M_X(v) = \frac{2}{2-v}$. Find the mean, mean square and variance?

Solution: We know that $m_n = \frac{d^n}{dv^n} M_X(v)\Big|_{v=0}$

1. $m_1 = \frac{d}{dv}\left[\frac{2}{2-v}\right]$

$$= 2\left[\frac{(2-v)(0) - (1)(-1)}{(2-v)^2}\right]$$

$$= 2\left[\frac{1}{(2-v)^2}\right]_{v=0}$$

$$= 2\left[\frac{1}{4}\right] = \frac{1}{2}$$

2. $m_2 = \frac{d^2}{dv^2}\left[\frac{2}{2-v}\right]$

$$= 2\frac{d}{dv}\left[\frac{1}{(2-v)^2}\right]$$

$$= 2\left[\frac{0 - (1)(2)(2-v)^3(-1)}{(2-v)^4}\right]_{v=0}$$

$$= \frac{8}{16} = \frac{1}{2}$$

3. Variance: $\sigma_X^2 = m_2 - m_1^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Another Method:

$$M_X(v) = E[e^{vX}] = \frac{2}{2-v} = \frac{1}{1-\left(\frac{v}{2}\right)} = \left(1 - \frac{v}{2}\right)^{-1}$$

We know that $(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$$E[e^{vX}] = \left(1 - \frac{v}{2}\right)^{-1}$$

$$E\left[1 + vX + \frac{v^2X^2}{2!} + \ldots\right] = 1 + \frac{v}{2} + \frac{v^2}{4} + \ldots$$

$$E[1] + E[vX] + E[\frac{v^2X^2}{2!}] + \ldots = 1 + \frac{v}{2} + \frac{v^2}{4} + \ldots$$

$$1 + vE[X] + \frac{v^2}{2}E[X^2] + \ldots = 1 + \frac{v}{2} + \frac{v^2}{4} + \ldots$$

From the above equation, equate the co-efficients

$$vE[X] = \frac{v}{2} \Rightarrow E[X] = \frac{1}{2} \longrightarrow m_1$$

$$\frac{v^2}{2}E[X^2] = \frac{v^2}{4} \Rightarrow E[X^2] = \frac{1}{2} \longrightarrow m_2$$

$$\frac{v^2}{2}E[X^3] = \frac{v^2}{8} \Rightarrow E[X^3] = \frac{1}{8} \longrightarrow m_3$$

Variance $= m_2 - m_1^2 = \frac{1}{4}$

Problem: A random variavle 'X' has PDF $f_X(x) = \frac{1}{2^x};\ x = 1, 2, 3, \ldots$ Find MGF, mean, mean square, and variance.

$$M_X(v) = \int_{-\infty}^{\infty} f_X(x)e^{vx}\,dx$$

$$= \sum_{x=1}^{\infty} \frac{1}{2^x} e^{vx}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^v}{2}\right)^x$$

$$= 1 + \sum_{x=1}^{\infty} \left(\frac{e^v}{2}\right)^x - 1$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^v}{2}\right)^x - 1$$

$$= \frac{1}{1 - \frac{e^v}{2}} - 1 = \frac{2 - 2 + e^v}{2 - e^v}$$

$$M_X(v) = \frac{e^v}{2 - e^v}$$

$$m_1 = \frac{d}{dv}\left[\frac{e^v}{2 - e^v}\right]_{v=0}$$

$$= \left[\frac{(2 - e^v)(e^v) - e^v(-e^v)}{(2 - e^v)^2}\right]_{v=0}$$

$$= \left[\frac{2e^v - e^{2v} + e^{2v}}{(2 - e^v)^2}\right]_{v=0}$$

$$= \left[\frac{2(1)}{(2-1)^2}\right]$$

$$= 2$$

$$m_2 = \frac{d^2}{dv^2}\left[\frac{e^v}{2 - e^v}\right]_{v=0}$$

$$= \frac{d}{dv}\left[\frac{(2 - e^v)(e^v) - e^v(-e^v)}{(2 - e^v)^2}\right]_{v=0}$$

$$= \frac{d}{dv}\left[\frac{2e^v - e^{2v} + e^{2v}}{(2 - e^v)^2}\right]_{v=0}$$

$$= \frac{d}{dv}\left[\frac{2e^v}{(2 - e^v)^2}\right]_{v=0}$$

$$= \left[\frac{(2 - e^v)^2(2e^v) - (2e^v)\,2(2 - e^v)(-e^v)}{(2 - e^v)^4}\right]_{v=0}$$

$$= \frac{(2-1)^2(2 \times 1) - (2 \times 1)\,2(2-1)(-1)}{(2-1)^4}$$

$$= \frac{2+4}{1} = 6$$

3. Variance: $\mu_2 = m_2 - m_1^2 = 6 - 2^2 = 2$

$$\boxed{\therefore\ m_1 = 2; \quad m_2 = 6; \quad \mu_2 = 2}$$

4.1.4 Properties of MGF

1. If MGF of r.v 'X' is $M_X(v)$ then r.v 'cX' is $M_{cX} = M_X(cv)$

   Proof.

   $$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x)e^{vx}\, dx$$

   $$M_{cX}(v) = \int_{x=-\infty}^{\infty} f_X(x)e^{v(cx)}\, dx$$

   $$= \int_{x=-\infty}^{\infty} f_X(x)e^{cvx}\, dx$$

   $$=; M_X(cv)$$

   □

2. If MGF of r.v 'X' is $M_X(v)$ then r.v '$aX + b$' is $M_{aX+b} = e^{bv}M_X(v)$

   Proof.

   $$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x)e^{vx}\, dx$$

   $$M_{aX+b}(v) = \int_{x=-\infty}^{\infty} f_X(x)e^{v(ax+b)}\, dx$$

   $$= e^{vb}\int_{x=-\infty}^{\infty} f_X(x)e^{avx}\, dx$$

   $$= e^{vb}\, M_X(av)$$

   □

3. If X and Y are independent r.v's then the MGF will be the product of two idivid-ual of MGF. i.e., $M_{X+Y}(v) = M_X(v)\, M_Y(v)$

   Proof.

   $$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x)e^{vx}\, dx$$

   $$M_{X+Y}(v) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_XY(x, y)\, e^{v(x+y)}\, dy\, dx \quad \because f_{XY}(x, y) = f_X(x)f_Y(y)$$

$$= \int_{x=-\infty}^{\infty} f_X(x)e^{vx}\,dx \int_{y=-\infty}^{\infty} f_Y(y)e^{yx}\,dy \qquad \because \text{Independent}$$

$$M_{X+Y}(v) = M_X(v)\,M_Y(v)$$

□

4. If MGF of r.v 'X' is $M_X(v)$ then r.v then $M_{\frac{X+a}{b}} = e^{\frac{va}{b}} M_X(\frac{v}{b})$

Proof.

$$M_X(v) = E[e^{vX}] = \int_{x=-\infty}^{\infty} f_X(x)e^{vx}\,dx$$

$$\overline{M_{\frac{X+a}{b}}} = \int_{x=-\infty}^{\infty} f_X(x)e^{v(\frac{x+a}{b})}\,dx$$

$$= e^{\frac{va}{b}} \int_{x=-\infty}^{\infty} f_X(x)e^{\frac{vx}{b}}\,dx$$

$$= e^{\frac{va}{b}} M_X(\frac{v}{b})$$

□

4.1.5 Conditional CDF and PDF

Let A and B are two events, conditional probability can be defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}; \quad P(B) \quad 0$$

Here, let event 'A' interms of continuous r.v $A = -\infty < X < \infty$ can be defined as $\{X \le x\}$ and conditionalprobability $P\{X \le x|B\}$.

$$P(X \le x|B) = \frac{P\ X \le xB}{P(B)} = \frac{P\ X \le x \cap B}{P(B)}; \quad P(B) \quad 0$$

- The conditional distribution (CDF) can be written as

$$F_X(x|B) = P(X \le x|B) = \frac{P\ X \le xB}{P(B)} = \frac{P\ X \le x \cap B}{P(B)}; \quad P(B) \quad 0$$

$P\ X \le\ x \cap B$ is called probability of joint event. The conditional distribution function my continuous or discrete.

- The conditional PDF can be obtained by derivating conditional CDF

$$f_X(x|B) = \frac{d}{dx}F_X(x|B)$$

- Similarly, if the conditional PDF is known then, the conditional CDF is

$$F_X(x|B) = P\left\{\frac{-\infty \leq X \leq x}{B}\right\} = \int_{-\infty}^{x} f_X(x|B)\,dx$$

4.1.5.1 Properties of conditional CDF function

1. The value of conditional CDF at $X = -\infty$ and $X = \infty$ is given by

$$F_X(x|B) = 0 \quad \text{at } x = -\infty; \quad \text{and} \quad F_X(x|B) = 1 \quad \text{at } x = \infty$$

2. $F_X(x|B)$ lies between 0 to 1.

$$0 \leq F_X(x|B) \leq 1$$

3. Conditional CDF is continuous and incrementing function. i.e.,

$$F_X(x|B) = F_X(x^+|B)$$

Proof. Let X is a r.v which takes the vaiable from $-\infty$ to ∞

$$F_X(x_2|B) > F_X(x_1|B); \qquad \text{if } x_2/B > x_1/B$$
$$F_X(x_2|B) = P\{-\infty \leq X \leq x_2|B\}$$
$$= P\{-\infty \leq X \leq x_1|B\} + P\{x_1|B \leq X \leq x_2|B\}$$
$$= P\{x_1|B\} + P\{x_1|BX \leq x_2|B\}$$
$$\therefore F_X(x_2|B) > P\{x_1|B\}; \qquad \text{if } x_2/B > x_1/B$$

So, $F_X(x|B)$ is a Non-decrementing function.

□

4. The conditional CDF between $x_1|B$ and $x_2|B$ can be written as

$$P\{x_1|B \leq X \leq x_2|B\} = F_X(x_2|B) - F_X(x_1|B); \quad x_2|B > x_1|B$$

4.1.5.2 Properties of conditional PDF function

1. It is non-negative function; $f_X(x|B) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x|B)\, dx = 1$
3. $F_X(x|B) = \int_{-\infty}^{x} f_X(u)\, du$
4. $P\{x_1|B \leq X \leq x_2|B\} = \int_{x_1}^{x_2} f_X(x|B)\, dx$

Problem: Two boxes contain red, green, and blue as shown in Table. Raandom variable represents selecting one ball from selected box. Probability selecting boxes $P(B_1) = \frac{2}{10}$ and $P(B_2) = \frac{8}{10}$.

X=x_i	Ball color	Ball		
		Box1	Box2	Total
$i = 1$	Red	5	80	85
$i = 2$	Green	35	60	95
$i = 3$	Blue	60	10	70
	Total	100	150	250

Find

1. $f_X(x|B_1)$ and $F_X(x|B_1)$
2. $f_X(x|B_2)$ and $F_X(x|B_2)$
3. (c) $f_X(x)$ and $F_X(x)$

Solution: Let 'X' be the discrete r.v which take values

$X = x_1 = 1$ represents selecting Red ball

$X = x_2 = 2$ represents selecting Green ball

$X = x_3 = 3$ represents selecting Blue ball

$f_X(x|B_1)$:

$P(X = 1|B_1) = \frac{5}{100}$

$P(X = 2|B_1) = \frac{35}{100}$

$P(X = 3|B_1) = \frac{60}{100}$

$f_X(x|B_2)$:

$P(X = 1|B_2) = \frac{80}{150}$

$P(X = 2|B_2) = \frac{60}{150}$

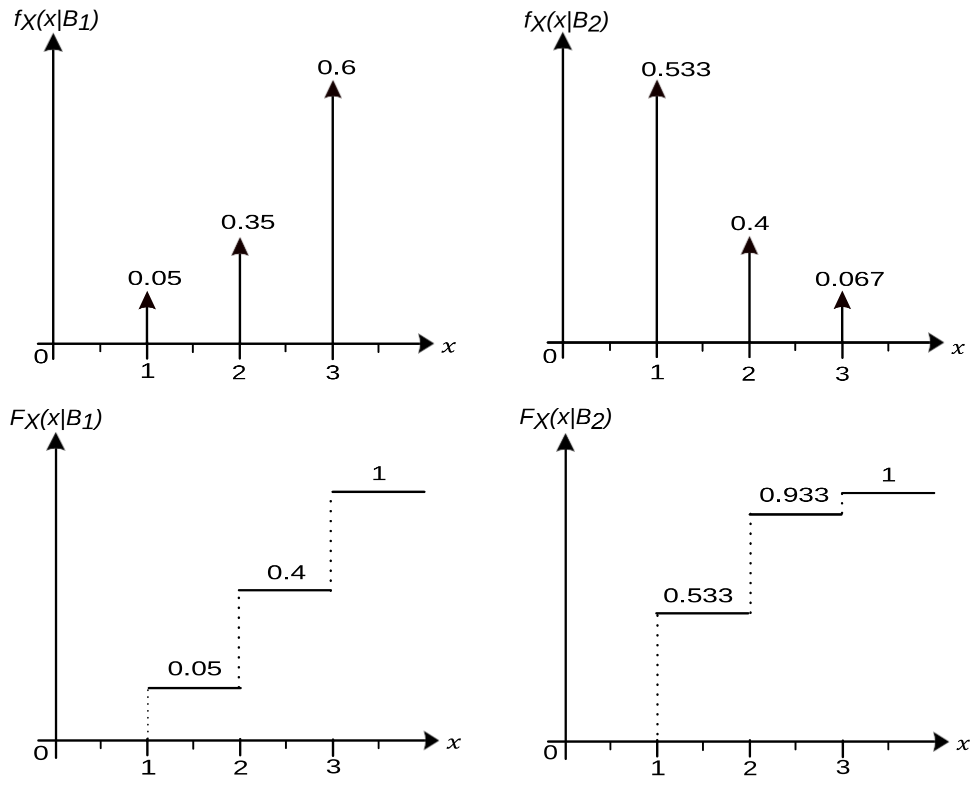
$P(X = 3|B_2) = \frac{10}{150}$

$$f_X(x|B_1) = \frac{5}{100}\delta(x-1) + \frac{35}{100}\delta(x-2) + \frac{60}{100}\delta(x-3)$$

$$f_X(x|B_2) = \frac{80}{150}\delta(x-1) + \frac{60}{150}\delta(x-2) + \frac{10}{150}\delta(x-3)$$

$$F_X(x|B_1) = \frac{5}{100}U(x-1) + \frac{35}{100}U(x-2) + \frac{60}{100}U(x-3)$$

$$F_X(x|B_2) = \frac{80}{150}U(x-1) + \frac{60}{150}U(x-2) + \frac{10}{150}U(x-3)$$



$$f_X(x|B) = \frac{P\{X \le x \cap B\}}{P\{B\}}$$

Given $P(B_1) = \frac{2}{10}$; $\quad P(B_2) = \frac{8}{10}$

$f_X(x_i = 1) = f_X(Red = x_1) = f_X(1) = p(X = 1)$

$$= P(X = 1|B_1)P(B_1) + P(X = 1|B_2)P(B_2)$$

$$= \frac{5}{100} \times \frac{2}{10} + \frac{80}{150} \times \frac{8}{10}$$

$$= 0.437$$

$f_X(x_i = 2) = f_X(Green = x_2) = f_X(2) = p(X = 2)$

$$= P(X = 2|B_1)P(B_1) + P(X = 2|B_2)P(B_2)$$

$$= \frac{35}{100} \times \frac{2}{10} + \frac{60}{150} \times \frac{8}{10}$$

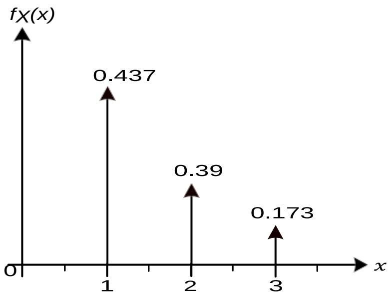
$$= 0.39$$

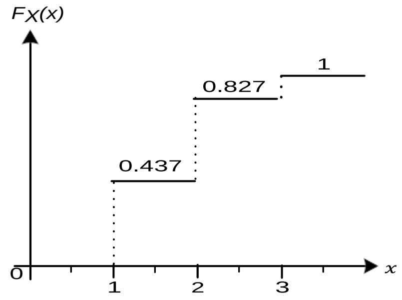
$f_X(x_i = 2) = f_X(Blue = x_3) = f_X(3) = p(X = 3)$

$$= P(X = 3|B_1)P(B_1) + P(X = 3|B_2)P(B_2)$$

$$= \frac{60}{100} \times \frac{2}{10} + \frac{10}{150} \times \frac{8}{10}$$

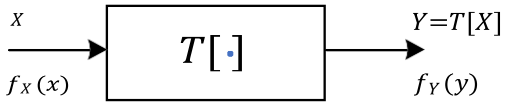
$$= 0.173$$





4.2 Transformation of a random variable

In many applications (practically) one random variable need to transformed to another random variable by performing some operation as shown in figure.



Here, X be the input r.v whose PDF is $f_X(x)$ and Y be the output r.v whose PDF is $f_Y(y)$. $T[\cdot]$ is the operation performed by system to transform X into Y, i.e., addition, subtraction, multiplication, square, integration etc.,

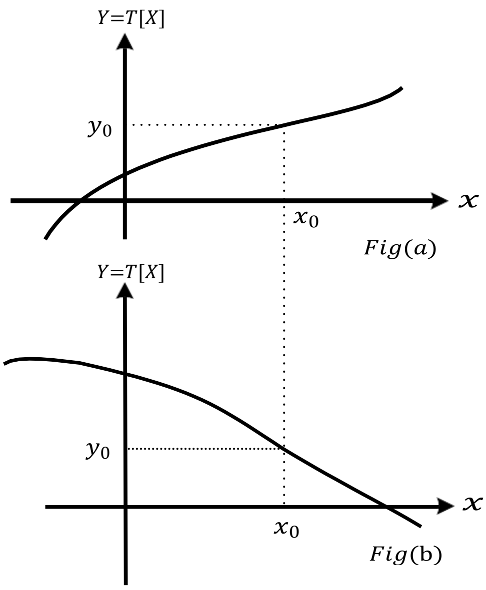
Types of transformation:

1. Monotonic Transformations of a continuous r.v
2. Non-Monotonic Transformations of a continuous r.v
3. Monotonic Transformations of a discrete r.v
4. Non-Monotonic Transformations of a discrete r.v

4.2.1 Continuous r.v, Monotonic transformation (increasing/descreasing)

Let 'X' is a continuous random variable and the transformation is said to be monotonic if one-to-one transformation between input and output random variable.

The transformation is said to be monotonically increasing if its satisfies the condition $T[X_2] > T[X_1]$; if $x_2 > x_1$ as shown in Fig. (a). The transformation is said to be monotonically decreasing if its satisfies the condition $T[X_2] < T[X_1]$; if $x_2 < x_1$ as shown in Fig.(b).



Let 'Y' have a particular value 'y_0' corresponding to the particular value x_0 if 'X' as shown Fig. (a)

$y_0 = T[x_0] \Rightarrow x_0 = T^{-1}[x_0]$. Here T^{-1} is inverse transform of 'T'.

Now the probability of the event $Y \leq y_0$ must be equal to probability the event $X \leq x_0$, because of the one-to-one correspondance between X and Y.

$$F_Y(y_0) = P\{Y \leq y_0\} = P\{X \leq x_0\} = F_X(x_0) = P\{-\infty \leq X \leq \infty\}$$

$$F_Y(y) = F_X(x)$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{x_0} f_X(x)\,dx$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{T^{-1}[y_0]} f_X(x)\,dx$$

By differentiating both sides with respect to 'y_0'

$$f_Y(y_0) = f_X\left[T^{-1}[y_0]\right] \cdot \frac{d}{dy}T^{-1}[y_0]$$

The above integration is is evaluated at particular point. In general,

$$f_Y(y) = f_X\left[T^{-1}[y]\right] \cdot \frac{d}{dy}T^{-1}[y]$$

$$\therefore f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$$

where $\frac{dy}{dx}$ is the slope of the transformation, it is positive and negative for monotonically increasing and decreasing respectively. But the PDF should always positive. Hence the above equation can be written as

$$\therefore f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

Here, $\left|\frac{dx}{dy}\right|$ is the inverse slope of the transformation.

Problem: Consider two r.v's X and Y, such that $Y = 2X+3$. The density function of r.v 'X' is shown in Fig.

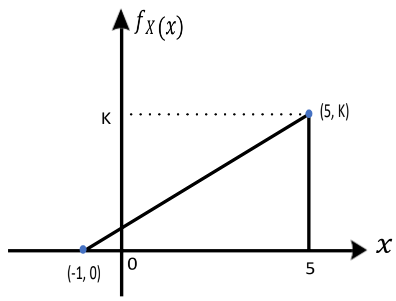
$$(x - x_1)(y_2 - y_1) = (y - y_1)(x_2 - x_1)$$

$$(x+1)(K-0) = (y-0)(5+1)$$

$$Kx + K = 5y + y$$

$$K(x+1) = 6y$$

$$y = \frac{K(x+1)}{6}$$



$$\therefore f_X(x) = \begin{cases} \frac{K}{6}(x+1); & -1 \le X \le 5 \\ 0; & \text{otherwise} \end{cases}$$

Total probability is unity.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\frac{K}{6}\int_{-1}^{5}(x+1)\, dx = 1$$

$$\frac{K}{6}\left[\frac{x^2}{2} + x\right]_{-1}^{5} = 1$$

$$\Rightarrow K = \frac{1}{3}$$

$$\therefore f_X(x) = \begin{cases} \frac{x+1}{18}; & -1 \le X \le 5 \\ 0; & \text{otherwise} \end{cases} \implies (1)$$

(ii) $f_Y(y) = f_X(x).\frac{dx}{dy}$ $\Rightarrow$ (2)

Given

$$y = 2x + 3 \Rightarrow x = \frac{y-3}{2}$$

$$\frac{dy}{dx} = 2 \Rightarrow \frac{dx}{dy} = \frac{1}{2}$$

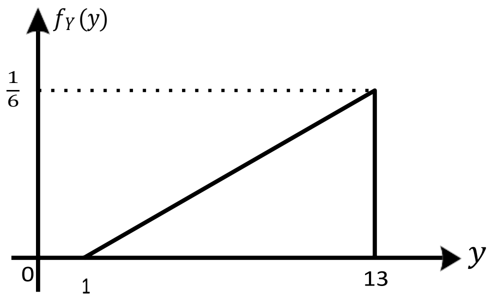
limits if $x = -1 \Rightarrow y = 1;$

and if $x = 5 \Rightarrow y = 13;$

$\Rightarrow$ (1) and (2)

$$\Rightarrow f_Y(y) = \frac{\frac{y-3}{2}+1}{18} \times \frac{1}{2} = \frac{y-1}{72}$$

$$\therefore f_Y(y) = \begin{cases} \frac{y-1}{72}; & 1 \le y \le 13 \\ 0; & \text{other wise} \end{cases}$$



for $y = 1 \Rightarrow f_Y(y) = 0$; for $y = 13 \Rightarrow f_Y(y) = \frac{1}{6}$

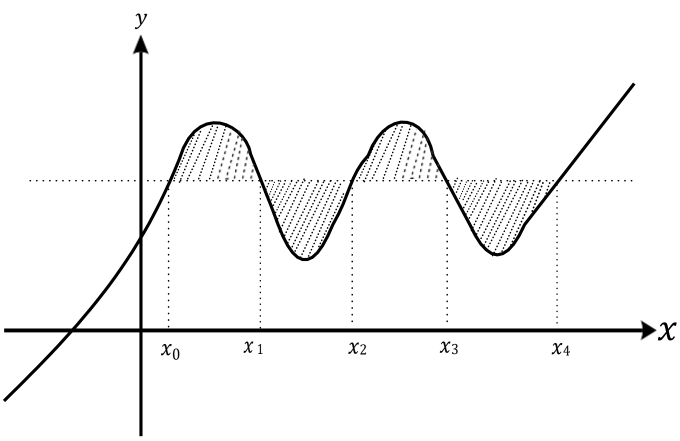
Total area under the curve:

$$\int_{y=1}^{13} \frac{y-1}{72}\, dy = \frac{1}{72}\left[\frac{(y-1)^2}{2}\right]_1^{13}$$

$$= \frac{1}{72}\left[\frac{244}{2}\right] = 1$$

4.2.2 Continuous r.v, Non-Monotonic transformation

Let us consider non-monotonic transfermation or many-to-one transfarmation in which input random variable to output random variable as shown in Fig.



Here the relationship between input and output PDF is given by

$$f_Y(y) = \sum_n f_X(x_n)\left|\frac{dx_n}{dy}\right|; \qquad \text{where } \left|\frac{dx_n}{dy}\right| \text{ is inverse of slope at all intervals.}$$

4.2.3 Discrete r.v, Monotonic transformation

If 'X' is a discrete random variable, whose PDF is $f_X(x)$ and CDF is $F_X(x)$, such taht $Y = T[X]$, whose PDF is $f_Y(y)$ and CDF is $F_Y(y)$ then

$$f_X(y) = \sum_n P(X = x_n)\, \delta(x - x_n) \qquad f_Y(y) = \sum_n P(Y = y_n)\, \delta(y - y_n)$$

$$F_X(y) = \sum_n P(X = x_n)\ U(x - x_n) \qquad f_Y(y) = \sum_n P(Y = y_n)\ U(y - y_n)$$

It is a one-to-one correspondance between X and Y, so that a set $\{y_n\}$ corresponeded to the set $\{x_n\}$ through the equation $Y_n = T[X_n]$.

∴ The probability $P[y_n]$ is equal to $P[X_n]$. Thus,

$$Y_n = T[X_n]$$

$$P[Y = y_n] = P[X = x_n]$$

4.2.4 Discrete r.v, Non-Monotonic transformation

If 'X' is discrete r.v and 'T' is not monotonic, the above procedure remains valid except there now exists probability that more than one value 'x_n' corresponds to a value y_n. In such case $P[y_n]$ will equal to the sum of the probabilities of the various x_n for which $Y_n = T[X_n]$.

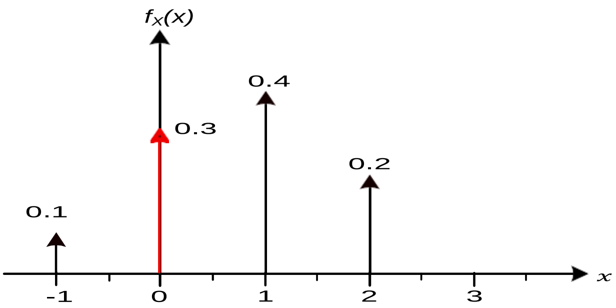
Problem: Let a discrete r.v 'X' has values $x = -1, 0, 1,$ and 2 wit probabilities $0.1, 0.3, 0.4$ and 0.2. The r.v 'X' is transformed to (a) $Y = 2X$ (b) $Y = 2 - x^2 + \frac{x^3}{3}$ then find $f_Y(y)$ and $F_Y(y)$?

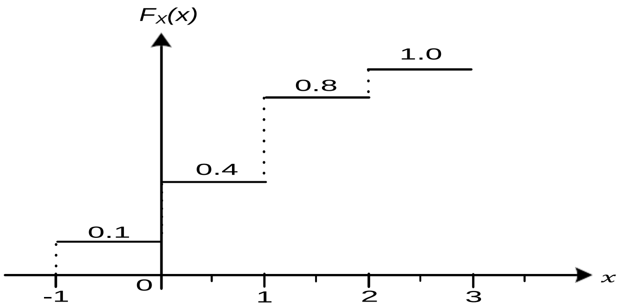
Solution:

$X = x_i$	-1	0	1	2
$P(X = x_i)$	0.1	0.3	0.4	0.2

$$f_X(x) = 0.1\delta(x+1) + 0.3\delta(x) + 0.4\delta(x-1) + 0.2\delta(x-2)$$

$$F_X(x) = 0.1U(x+1) + 0.3U(x) + 0.4U(x-1) + 0.2U(x-2)$$





Tranformation $Y = 2X$; $P(Y = y_n) = P(X = x_n)$

$x = -1 \Rightarrow y = -2; f_Y(-2) = P(Y = -2) = 0.1 = P(X = -1)$

$x = 0 \Rightarrow y = 0; f_Y(0) = P(Y = 0) = 0.3 = P(X = 0)$

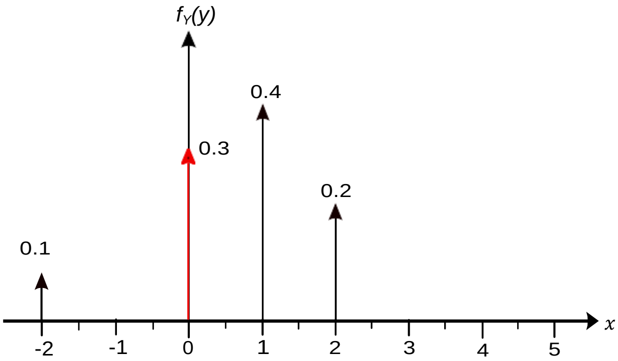
$x = 1 \Rightarrow y = 2; f_Y(2) = P(Y = 2) = 0.4 = P(X = 1)$

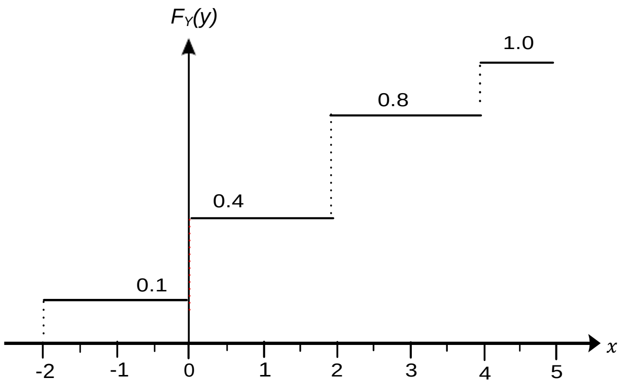
$x = 1 \Rightarrow y = 4; f_Y(4) = P(Y = 4) = 0.2 = P(X = 2)$

$Y = y_i$	-2	0	2	4
$P(Y = y_i)$	0.1	0.3	0.4	0.2

$f_Y(y) = 0.1\delta(y+2) + 0.3\delta(y) + 0.4\delta(y-2) + 0.2\delta(y-4)$

$F_Y(y) = 0.1U(y+2) + 0.3U(y) + 0.4U(y-2) + 0.2U(y-4)$





(b) Given $Y = 2 - x^2 + \frac{x^3}{3}$

$x = -1 \Rightarrow y = 2 - 1 - \frac{1}{3} = \frac{6 - 3 - 1}{3} = \frac{2}{3}$;

$x = 0 \Rightarrow y = 2$

$x = 1 \Rightarrow y = 2 - 1 + \frac{1}{3} = \frac{6 - 3 + 1}{3} = \frac{4}{3}$;

$x = 2 \Rightarrow y = 2 - 4 + \frac{8}{3} = \frac{6 - 12 + 18}{3} = \frac{2}{3}$;

$P(Y = y_n) = P(X = x_n)$; $f_X(x) = f_Y(y)$

$$f_Y\left(\frac{2}{3}\right) = P\left(Y = \frac{2}{3}\right) = P(X = -1) = 0.1$$

$$f_Y(2) = P(Y = 2) = P(X = 0) = 0.3$$

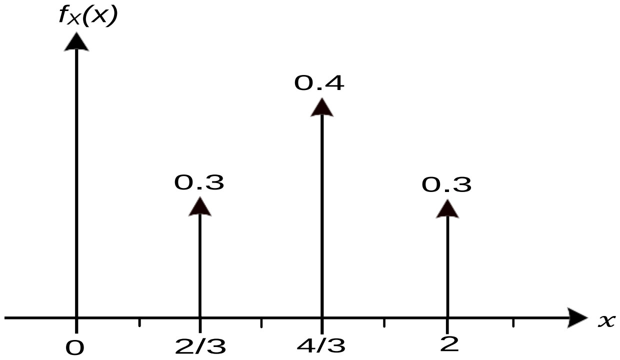
$$f_Y\left(\frac{4}{3}\right) = P\left(Y = \frac{4}{3}\right) = P(X = 1) = 0.4$$

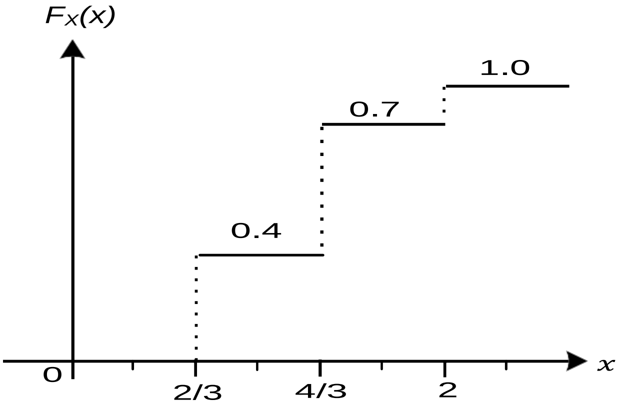
$$f_Y\left(\frac{2}{3}\right) = P\left(Y = \frac{2}{3}\right) = P(X = 2) = 0.2$$

$$f_Y(y) = 0.1\delta\left(y - \frac{2}{3}\right) + 0.3\delta(y - 2) + 0.4\delta\left(y - \frac{4}{3}\right) + 0.2\delta\left(y - \frac{2}{3}\right)$$

$$= 0.3\delta\left(y - \frac{2}{3}\right) + 0.4\delta\left(y - \frac{4}{3}\right) + 0.3\delta(y - 2)$$

$$F_Y(y) = 0.3U\left(y - \frac{2}{3}\right) + 0.4U\left(y - \frac{4}{3}\right) + 0.3U(y - 2)$$





Problem: A r.v 'X' having the values $-4, 1, 2, 3, 4$ having equal probabilities is $\frac{1}{5}$

(i) Find and plot $f_X(x)$, $F_X(x)$, mean and variance.

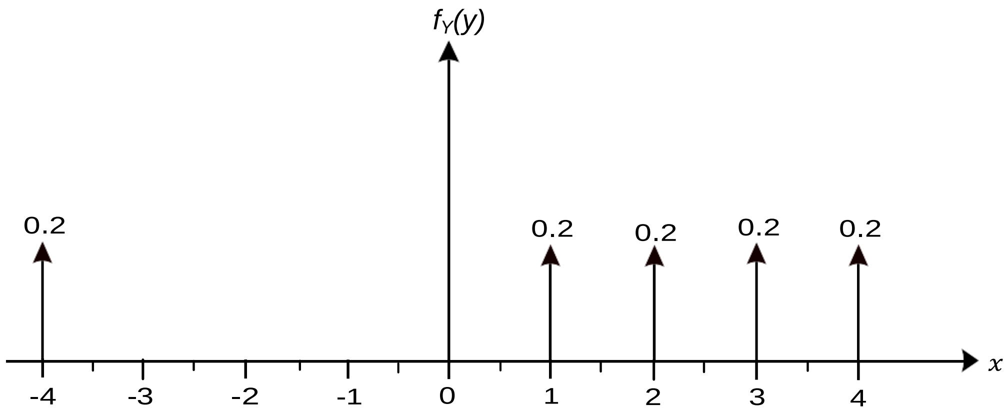
(ii) If $Y = x^3$; Find and plot $F_Y(y)$, $F_Y(y)$, mean and variance.

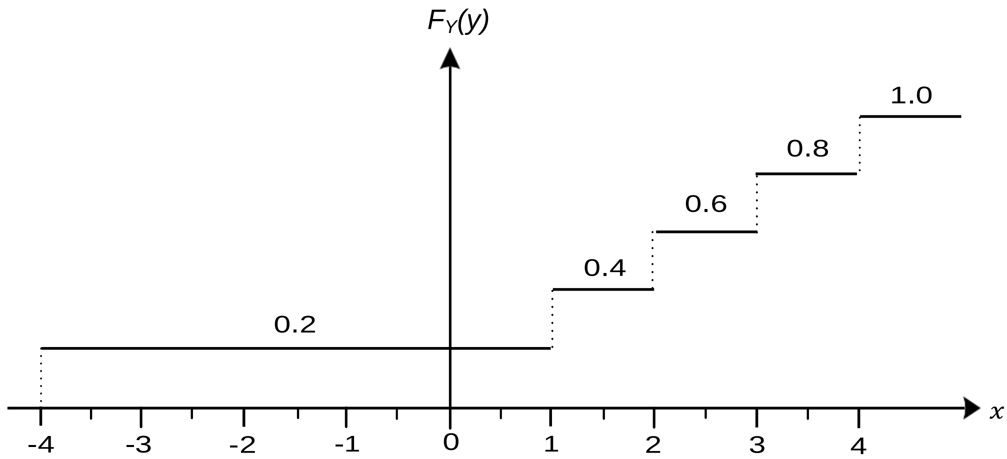
Solution: Given $X = \{-4, 1, 2, 3, 4\}$; $P(X) = \frac{1}{5} = 0.2$

i.e., $P(X = -4) = P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = 0.2$

$$f_X(x) = 0.2\delta(x + 4) + 0.2\delta(x - 1) + 0.2\delta(x - 2) + 0.2\delta(x - 3) + 0.2\delta(x - 4)$$

$$F_X(x) = 0.2U(x + 4) + 0.2U(x - 1) + 0.2U(x - 2) + 0.2U(x - 3) + 0.2U(x - 4)$$





$$E(X) = m_1 = \sum_{x_i} x_i\, f_X(x_i)$$

$$= -4(0.2) + 1(0.2) + 2(0.2) + 3(0.2) + 4(0.2)$$

$$= 0.2 \ -4 + 1 + 2 + 3 + 4$$

$$= 0.2 \times 0.6 = 1.2$$

$$E(X^2) = m_2 = \sum_{x_i} x_i^2\, f_X(x_i)$$

$$= (-4)^2(0.2) + (1)^2 0.2 + (2)^2 0.2 + (3)^2 0.2 + (4)^2 0.2$$

$$= 0.2 \ 16 + 1 + 4 + 9 + 16$$

$$= 0.2 \times 46 = 9.2$$

variance: $\sigma_X^2 = m_2 - m_1^2 = 9.2 - (1.2)^2 = 7.76$

(ii) Given $Y = x^3$; $\quad P(Y = y_i) = P(X = x_i)$

$x = -4 \Rightarrow y = (-4)^3 = -64; \quad P(Y = -64) = 0.2$

$x = 1 \Rightarrow y = (1)^3 = 1; \quad P(Y = 1) = 0.2$

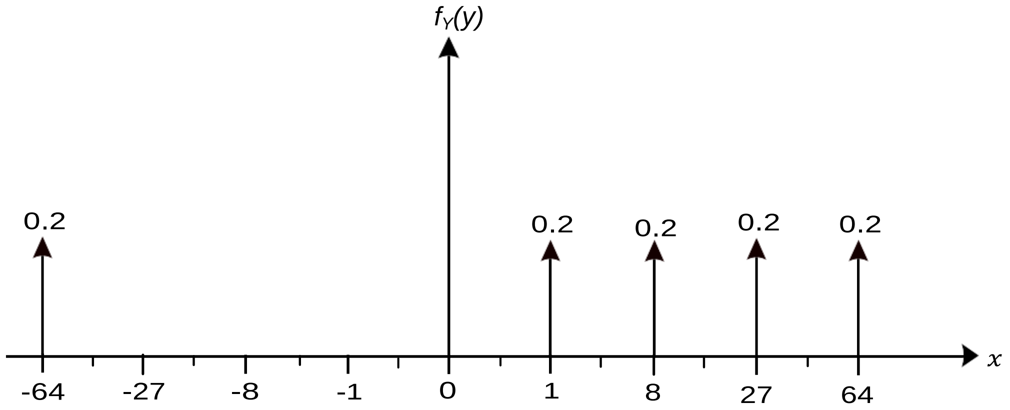
$x = 2 \Rightarrow y = (2)^3 = 8; \quad P(Y = 8) = 0.2$

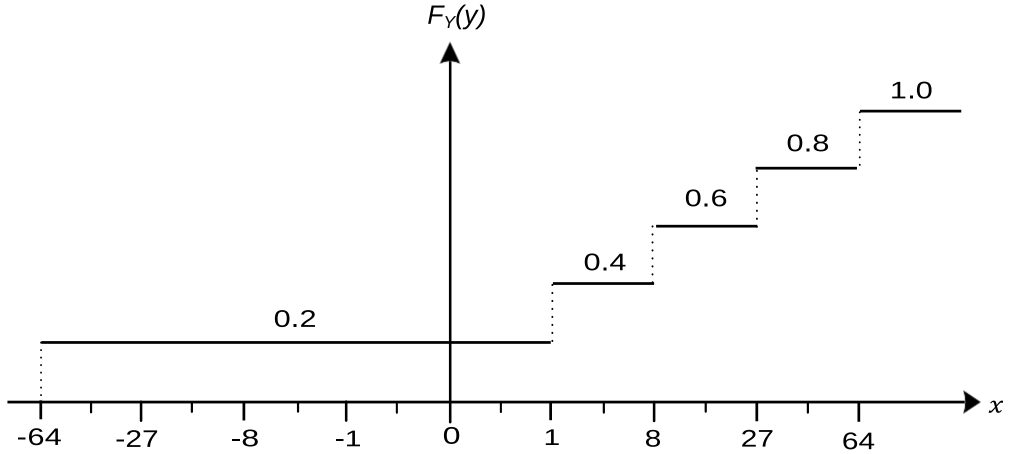
$x = 3 \Rightarrow y = (3)^3 = 27; \quad P(Y = 27) = 0.2$

$x = 4 \Rightarrow y = (4)^3 = 64; \quad P(Y = 64) = 0.2$

$f_Y(y) = 0.2\delta(y + 64) + 0.2\delta(y - 1) + 0.2\delta(x - 8) + 0.2\delta(y - 27) + 0.2\delta(y - 64)$

$f_Y(y) = 0.2U(y + 64) + 0.2U(y - 1) + 0.2U(x - 8) + 0.2U(y - 27) + 0.2U(y - 64)$





$$E(Y) = m_1 = \sum_{y_i} y_i f_Y(y_i)$$

$= (-64)(0.2) + (1)(0.2) + (8)(0.2) + (27)(0.2) + (64)(0.2)$

$= 0.2 \;\; 1 + 8 + 27$

$= 0.2 \times 36 = 7.2$

$$E(Y^2) = m_2 = \sum_{y_i} y_i^2 f_Y(y_i)$$

$= (-64)^2(0.2) + (1)^2(0.2) + (8)^2(0.2) + (27)^2(0.2) + (64)^2(0.2)$

$= 0.2 \;\; 64^2 + 1^2 + 8^2 + 27^2 + 64^2$

$= 0.2 \times 8986 = 1797.2$

Variance: $\sigma_Y^2 = m_2 - m_1^2 = 1797.2 - 7.2^2 = 1745.36$

4.3 Methods of defining Conditional events

The conditional distribution of random variable 'X' is

$$F_X(x|B) = P\{X \leq x|B\} \longrightarrow (1)$$

Let the event B in equation (1) be defined as $B = X \leq b$ where b is some real number $-\infty < b < \infty$

$$F_X(x|B) = P\{X \leq x|B\}$$
$$= P\{(X \leq x)|(X \leq b)\}$$
$$= \frac{P\{(X \leq x) \cap (X \leq b)\}}{P(X \leq b)}; \quad \longrightarrow (2)$$

$F_X(x|B)$ where $P(X \leq b) \neq 0$

Two cases to be considered to obtain $F_X(x|B)$

Case: (i) for $x \geq b$

$$P\{(X \leq x) \cap (X \leq b)\} = P\{X \leq x\}$$

since, $X \geq b$ means $(X \leq b) \subset X \leq x$. So,

$$\{(X \leq x) \cap (X \leq b)\} = \{X \leq x\}$$

substitute in eqn. (2)

$$\therefore F_X(x|B) = \frac{P(X)}{P(X \leq b)} = 1 \quad \text{for } x \geq b \longrightarrow (3)$$

Case: (ii) for $x \leq b$

Since $x < b$, means $(X \leq x) \subset (X \leq x)$. So,

$$\{(X \leq x) \cap (X \leq b)\} = \{X \leq x\}$$

$$P\{(X \leq x) \cap (X \leq b)\} = P\{X \leq x\}$$

substitute in eqn. (2), we get

$$F_X(x|B) = \frac{P\{X \leq x\}}{P\{X \leq b\}} \quad \because F_X(x) = P(X \leq x)$$

$$F_X(x|B) = \frac{F_X(x)}{F_X(b)} \quad \longrightarrow (4)$$

From equation (4) and (3), the conditional distribution function is defined as

$$F_X(x|B) = \begin{cases} \frac{F_X(x)}{F_X(b)}; & x < b \\ 1 & x \geq b \end{cases} \Longrightarrow (5)$$

The conditional density function is obtained by differentiating equation (5) with respect to 'x', ni.e., $f_X(x|B) = \frac{d}{dx} F_X(x|B)$ we get

$$f_X(x|B) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx}; & x < b \\ 0 & x \geq b \end{cases} \Longrightarrow (6)$$

where $\frac{d}{dx} F_X(b) = \int_{-\infty}^{b} f_X(x)\,dx.$

4.3.1 Conditioning a continuous random variable

In an experiment that produces a random variable X, there are occasions in which we can not observe X. Instead , we obtain information about X without learning its precise value.

EX: The experiment in which you wait for professor to arrive the probability lecture. Let X denote the arrival time in minutes either before $(X < 0)$ or after $(X > 0)$ the scheduled lecture time. when you observe that the professor is already two minutes late but has not arrived, you have learned that $X > 2$ but you have not learned the precise value of X.

4.3.1.1 Conditional PDF given an event

Definition: For a random variable 'X' with PDF is $f_X(X)$ and event $B \subset S_X$ with $P[B] > 0$, the conditional PDF of 'X' given B is

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]}; & x \in B \\ 0; & \text{otherwise} \end{cases}$$

Problem: Suppose the duration T (in minutes) of telephone call is an exponentional random variable.

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-\frac{t}{3}}; & t \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

For calls that atleast 2 minutes, what is the conditional probability of the call duration?

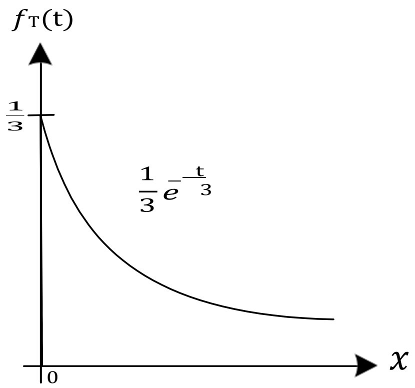
Solution:In this case, the conditioning event $T > 2$. The probability of the event

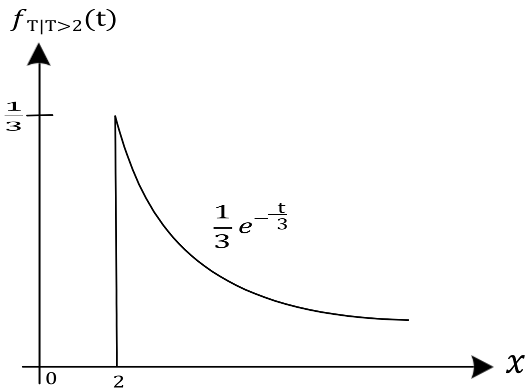
$$P(T > 2) = \int_{2}^{\infty} f_T(t)\,dt = e^{-\frac{2}{3}}$$

The conditional probability of T given $T > 2$ is

$$f_{T|T>2}(t) = \begin{cases} \dfrac{f_T(t)}{P(T>2)} & t > 2 \\ 0; & \text{otherwise} \end{cases}$$

$$\therefore f_{T|T>2}(t) = \begin{cases} \frac{1}{3}e^{-\frac{(t-2)}{3}}; & t > 2 \\ 0; & \text{otherwise} \end{cases}$$





Note: $f_{T|T>2}(t)$ is a time shifted version of $f_T(t)$

An interpretation of this result is that if the call is in progress after 2 minutes, the duration of the call is 2 minutes plus an exponentional time equal to the duration of new call.

4.3.1.2 Conditional expected value given an event

- If $\{x \in B\}$, the conditional expected value of X is

$$E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x)\ dx$$

- The conditional expected value of $g(x)$ is

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x)\ dx$$

- The conditional variance is

$$Var(X|B) = E\left[(X - \overline{X}_{X|B})^2|B\right] = E[X^2|B] - \overline{X}_{X|B}^2$$

- The conditional standard deviation: $\sigma_{X|B} = \sqrt{Var(X|B)}$

Note: Conditional variance and standard deviation are useful because they measure the spread of the random variable after we learn the conditioning information B. If the conditional standard deviation $\sigma_{X|B}$ is much smaller than σ_X, then that we learningthe occurance of B reduces our uncertannity about X because it shrinks the range of typical values of X.

CHAPTER 5

Multiple Random Variables

5.1 Vectors (or) Multiple random Variables

In many engineering applications situations arises where it is necessary to make use of more than one variable, say two r.v or several r.v.

Consider a sample space 'S', let X and Y are two r.v on it. Let the specific values of X and Y are denoted by x and y respectively then any ordered pair of numbers (x, y) is considered to be a random point in the xy-plane. This random point may be taken as a specific value of a vector random variable or a random vector. The figure shows the mapping involved in going from sample space 'S' to the xy-plane. Here S_J is joint sample space.

In a more general case where 'N' random variables $X_1, X_2, X_3 ... X_N$ are defined on a sample space 'S'. We call the r.vs to be components of an N-dimensional random vector (or) N-dimensional random variable.

5.2 Joint Distribution

Let us consider two events A is a function of 'x' and B is function of 'y' such that $A = \{X \leq x\}$ and $B = \{Y \leq y\}$

The joint event $A \cap B = \{X \leq x\}$ and $\{Y \leq y\}$ are shown in Fig.

The probability of two events A and B is called CDF or distribution function which can be written as

$$F_X(x) = P\{X \leq x\} \qquad F_Y(y) = P\{Y \leq y\}$$

The probability of joint event $\{X \leq x$ and $Y \leq y\}$, which is function of X and Y is called joint CDF (or) PDF is denoted $F_{XY}(x, y)$ and $f_{XY}(xy)$.

'X' r.v $\longrightarrow f_X(x)$ is PDF $\longrightarrow F_X(x) = P\{-\infty \leq X \leq x\} = \int_{u=-\infty}^{\infty} f_X(u)\, du$

'Y' r.v $\longrightarrow f_Y(y)$ is PDF $\longrightarrow F_Y(y) = P\{-\infty \leq Y \leq y\} = \int_{v=-\infty}^{\infty} f_X(v)\, dv$

$F_XY(x, y) = P\{X \le x, Y \le y\} = P\{-\infty \le X \le \infty\}$

$$= \underbrace{\int_{u=-\infty}^{\infty}\int_{v=-\infty}^{\infty} f_{XY}(u, y)}_{\text{Joint PDF}}\, dv\, du = \underbrace{F_{XY}(x, y)}_{\text{Joint CDF}}$$

If we know $F_X(x, y)$ then $f_X(x) = \frac{d}{dx} F_X(x) = \frac{\partial^2}{\partial x \partial y} F_X(x, y)$

Similarly if we know $F_{XY}(x, y)$ then $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$

5.2.1 Joint probability density function

Let X and Y are two r.v, joint PDF can be written as $f_{XY}(x, y) = P(X = x, Y = y)$. Joint PDF can ve obtained by evaluating second derivative of joint distribution function i.e.,

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

For 'N' random variables, the joint PDF can be written as

$$f_{XY}(x, y) = \frac{\partial^4 F_{X_1 X_2 X_3 \ldots X_N}(x_1, x_2, x_3, \ldots x_N)}{\partial x_1\, \partial x_2\, \partial x_3\, \partial \ldots x_N}$$

5.2.2 Properties of Joint PDF: $f_{XY}(x, y)$

1. $f_{XY}(x, y)$ is a non-negative i.e., $f_{XY}(x, y) \ge 0$

2. $o \le f_{XY}(x, y) \le 1$

3. Total probability: $\int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2} f_{XY}(x, y)\, dy\, dx$

4.
$$P\{(x_1 \le X \le x_2) \cap (y_1 \le Y \le y_2)\}$$
$$= \int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2} f_{XY}(x, y)\, dy\, dx$$
$$= F_{XY}(x_1, y_1) + F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1)$$

5. Marginal PDF and CDF of 'X'

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy \text{ (or) } f_X(x) = \frac{d}{dx} F_X(x)$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x, y)\, dx \text{ (or) } f_Y(y) = \frac{d}{F_X(x)}\, dy$$

6. The Joint PDF: $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$

5.2.3 Properties of Joint CDF: $F_{XY}(x, y)$

1.
$$
\begin{aligned}
F_{XY}(x, y) &= P\{(-\infty \le X \le x) \cap (-\infty \le Y \le y)\} \\
&= P\{X \le x \cap Y \le y\} \\
&= \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f_{XY}(u, v)\, dv\, du
\end{aligned}
$$

2. $F_{XY}(-\infty, -\infty) = F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$

3. $F_{XY}(+\infty, +\infty) = 1$

4. $F_{XY}(+\infty, y) = F_Y(y)$; $F_{XY}(x, +\infty) = F_X(x)$

5. $F_{XY}(x, y)$ is a non-decreasing function.

6. $F_{XY}(x, y)$ is a continuous function.

Problem 1: Let X and Y be the continuous r.v with Joint PDF is given by

$$f_X(x) = \begin{cases} b\,e^{-x}\cos y; & 0 \le x \le 2 \text{ and } 0 \le y \le \frac{\pi}{2} \\ 0; & \text{Else where} \end{cases}$$

1. Find constant 'b'
2. Find and plot $f_X(x)$ and $F_X(x)$
3. Find and plot $f_Y(y)$ and $F_Y(y)$
4. Find $F_{XY}(x, y)$
5. Find $P\{(0 \le X \le 1) \cap (0 \le Y \le \frac{\pi}{4})\}$

Solution:

1. Total probability = 1

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx = 1$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{\frac{\pi}{2}} be^{-x}\cos y\, dy\, dx = 1$$

$$\Rightarrow \int_{x=0}^{2} be^{-x}\left[\int_{y=0}^{\frac{\pi}{2}} \cos y\, dy\right] dx = 1$$

$$\Rightarrow \int_{x=0}^{2} be^{-x}\left[\sin y\right]_0^{\frac{\pi}{2}} dx = 1$$

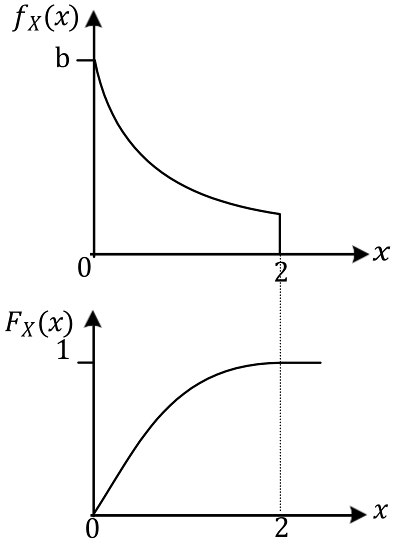
$$\Rightarrow \int_{x=0}^{2} be^{-x}\left[1 - 0\right] dx = 1$$

$$\Rightarrow b\left[\frac{e^{-x}}{-1}\right]_0^2 = 1$$

$$\Rightarrow b\left[\frac{e^{-2}}{-1} - \frac{e^{0}}{-1}\right] = 1$$

$$\Rightarrow b\left[-e^{-2} + 1\right] = 1$$

$$\boxed{b = \frac{1}{1 - e^{-2}} = 1.1565}$$



2. Marginal PDF $f_X(x)$:

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy$$

$$= \int_{y=0}^{\frac{\pi}{2}} be^{-x}\cos y\, dy$$

$$= be^{-x}\left[\sin y\right]_0^{\frac{\pi}{2}} = be^{-x}$$

$$\therefore f_X(x) = \begin{cases} b\,e^{-x}\cos y; & 0 \le x \le 2 \\ 0; & \text{Else where} \end{cases}$$

2. Marginal CDF $F_X(x)$:

The given intervals

$-\infty \le x \le 0; \quad 0 \le x \le 2; \quad x \ge 2$

Case I : $-\infty \le X \le 0$

$$F_X(x) = P\{-\infty \le X \le x\}$$

$$= \int_{u=-\infty}^{x} f_X(u)\, du$$

$$= \int_{-\infty}^{0} 0\, du = 0$$

$$\therefore F_X(x) = 0; \quad -\infty \le x \le 0$$

Case II : $0 \le X \le 2$

$$F_X(x) = P\{-\infty \le X \le x\}$$

$$= \int_{u=-\infty}^{0} \overset{=0}{f_X(u)\, du} + \int_{u=0}^{x} f_X(u)\, du$$

$$= \int_{u=0}^{x} b\, e^{-u}\, du = b \left[\frac{e^{-u}}{-1}\right]_0^x$$

$$= b\left[1 - e^{-x}\right]$$

$$\therefore F_X(x) = b\left[1 - e^{-x}\right]; \quad 0 \le x \le 2$$

Case III : $X \ge 2$

$$F_X(x) = P\{-\infty \le X \le x\}$$

$$= \int_{u=-\infty}^{0} \overset{=0}{f_X(x)\, dx} + \int_{x=0}^{2} f_X(x)\, dx + \int_{u=2}^{x} \overset{=0}{f_X(u)\, du}$$

$$= \int_{u=0}^{2} b\, e^{-u}\, du$$

$$= b\left[\frac{e^{-u}}{-1}\right]_0^2$$

$$= b\left[1 - e^{-2}\right] = \frac{1}{1-e^{-2}}\left[1 - e^{-2}\right]$$

$$= 1$$

$$\therefore F_X(x) = 1; \quad 0 \le x \le 2$$

$$\therefore F_X(x) = \begin{cases} 0; & -\infty \le x \le 0 \\ b\,[1 - e^{-x}]; & 0 \le x \le 2 \\ 1; & x \ge 2 \end{cases}$$

3. Marginal PDF $f_Y(y)$:

$$f_Y(y) = \int_{x=-\infty}^{x} f_{XY}(x, y)\,dx$$

$$= \int_{x=0}^{2} be^{-x}\cos y\,dx$$

$$= b\cos y\left[\frac{e^{-x}}{-1}\right]_0^2$$

$$= b\cos y\,[1 - e^{-2}]$$

$$= \frac{1}{1 - e^{-2}} \times \cos y\,[1 - e^{-2}]$$

$$= \cos y$$

$$\therefore f_X(x) = \begin{cases} \cos y & 0 \le x \le 2 \\ 0; & \text{Else where} \end{cases}$$

Marginal CDF $F_Y(y)$:

Given intervals are $-\infty \le y \le 0;\quad 0 \le y \le \frac{\pi}{2};\quad y \ge \frac{\pi}{2}$

Case I : $-\infty \le X \le 0$

$$F_Y(y) = P\{-\infty \le Y \le y\}$$

$$= \int_{u=-\infty}^{x} f_X(v)\,dv$$

$$= \int_{-\infty}^{0} 0\,dv = 0$$

$$\therefore F_Y(y) = 0; \quad -\infty \le y \le 0$$

Case II : $0 \le y \le 2$

$$F_Y(y) = P\{-\infty \le Y \le y\}$$

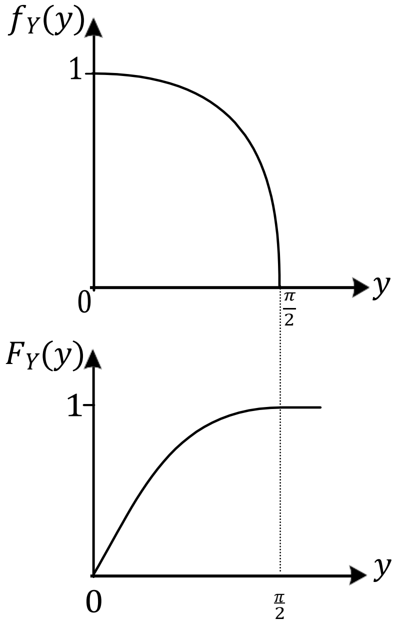
$$= \int_{y=-\infty}^{0} f_Y(y)\,dy \;(=0) + \int_{v=0}^{y} f_Y(v)\,dv$$

$$= \int_{v=0}^{y} \cos v\,dv$$

$$= \left[\sin v\right]_0^y$$

$$= \sin y$$

$$\therefore F_X(x) = \sin y; \quad 0 \le x \le \frac{\pi}{2}$$



Case III : $X \ge \frac{\pi}{2}$

$$F_Y(y) = P\{-\infty \le Y \le x\}$$

$$= \int_{y=-\infty}^{0} f_Y(y)\,du \;(=0) + \int_{y=0}^{\frac{\pi}{2}} f_Y(y)\,dy + \int_{v=\frac{\pi}{2}}^{y} f_Y(v)\,dv \;(=0)$$

$$= \int_{u=0}^{\frac{\pi}{2}} b \cos y \, dy = \sin y \Big|_0^{\frac{\pi}{2}} = 1$$

$$\therefore F_Y(y) = 1; \quad 0 \le x \le \frac{\pi}{2}$$

$$\therefore F_Y(y) = \begin{cases} 0; & -\infty \le y \le 0 \\ \sin y; & 0 \le y \le \frac{\pi}{2} \\ 1; & y \ge \frac{\pi}{2} \end{cases}$$

4. To find $F_{XY}(x, y)$: There are three cases

(a) $-\infty \le x \le 0$ and $-\infty \le y \le 0$

(b) $0 \le x \le 2$ and $0 \le y \le \frac{\pi}{2}$

(c) $2 \le x \le \infty$ and $\frac{\pi}{2} \le y \le \infty$

Case (a): $-\infty \le x \le 0$ and $-\infty \le y \le 0$

$$F_{XY}(x, y) = P\{(-\infty \le X \le x) \cap (-\infty \le X \le x)\}$$

$$= \int_{u=-\infty}^{0} \int_{v=-\infty}^{0} f_{XY}(x, y) \, dv \, du = 0$$

$$\therefore F_{XY}(x, y) = 0; \quad -\infty \le x \le 0 \text{ and } -\infty \le y \le 0$$

Case (b): $0 \le x \le 2$ and $0 \le y \le \frac{\pi}{2}$

$$F_{XY}(x, y) = P\{(-\infty \le X \le x) \cap (-\infty \le X \le x)\}$$

$$= \int_{x=-\infty}^{0} \int_{y=-\infty}^{0} f_{XY}(x, y) \, dy \, dx + \int_{u=0}^{x} \int_{v=0}^{y} f_{XY}(x, y) \, dv \, du$$

$$= \int_{u=0}^{x} \int_{v=0}^{y} b \, e^{-u} \cos v \, dv \, du$$

$$= b \int_{u=0}^{x} e^{-u} \left[\int_{v=0}^{y} \cos v \, dv \right] du$$

$$= b \left[\sin v \right]_0^y \left[\frac{e^{-u}}{-1} \right]_0^x$$

$$= b \left[\sin y - 0 \right] \left[-e^{-x} + 1 \right]$$

$$\therefore F_{XY}(x, y) = b \left(1 - e^{-x}\right) \sin y; \quad 0 \le x \le 2 \text{ and } 0 \le y \le \frac{\pi}{2}$$

Case (c): $2 \le x \le \infty$ and $\frac{\pi}{2} \le y \le \infty$

$$F_{XY}(x, y) = P\{(-\infty \le X \le x) \cap (-\infty \le X \le x)\}$$

$$= \int_{x=-\infty}^{0}\int_{y=-\infty}^{0} f_{XY}(x, y)\, dy\, dx + \int_{x=0}^{2}\int_{y=0}^{\frac{\pi}{2}} f_{XY}(x, y)\, dy\, dx + \int_{u=2}^{x}\int_{v=0}^{\frac{\pi}{2}} f_{XY}(x, y)\, dv\, du$$

$$= \int_{x=0}^{2}\int_{y=0}^{\frac{\pi}{2}} b\, e^{-x} \cos y\, dy\, dx$$

$$= b \int_{x=0}^{x} e^{-x} \left[\int_{y=0}^{\frac{\pi}{2}} \cos y\, dy\right] dx$$

$$= b \left[\sin y\right]_0^{\frac{\pi}{2}} \left[\frac{e^{-x}}{-1}\right]_0^2$$

$$= b\,[1 - 0]\,[-e^{-2} + 1]$$

$$= \frac{1}{1 - e^{-2}} \times [1 - e^{-2}] = 1$$

$$\therefore F_{XY}(x, y) = 1; \quad 2 \le x \le \infty \text{ and } \frac{\pi}{2} \le y \le \infty$$

$$\therefore F_{XY}(x, y) = \begin{cases} 0; & x \le 0 \text{ and } y \le 0 \\ b\,(1 - e^{-x}) \sin y; & 0 \le x \le 2 \text{ and } 0 \le y \le \frac{\pi}{2} \\ 1; & x \ge 0 \text{ and } y \ge \frac{\pi}{2} \end{cases}$$

5.

$P\{(0 \le X \le 1) \text{ and } (0 \le Y \le \frac{\pi}{4})\}$

$$= \int_{x=0}^{1}\int_{y=0}^{\frac{\pi}{4}} b\, e^{-x} \cos y\, dy\, dx$$

$$= \int_{x=0}^{1} b\, e^{-x} \left[\sin x\right]_0^{\frac{\pi}{4}} dx$$

$$= \int_{x=0}^{1} b\, e^{-x} \left[\frac{1}{\sqrt{2}} - 0\right] dx$$

$$= \frac{b}{\sqrt{2}} \left[\frac{e^{-x}}{-1}\right]_0^1$$

$$= \frac{b}{\sqrt{2}} \left[\frac{e^{-1}}{-1} + 1\right]$$

$$= \frac{b}{\sqrt{2}} \times \left[1 - e^{-1}\right]$$

$$= \frac{1}{1 - e^{-2}} \times \frac{1}{\sqrt{2}} \times \left[1 - e^{-1}\right]$$

$$= 0.5169$$

Problem 2. Find $F_{XY}(x, y)$, $F_X(x)$ and $f_X(x)$, $F_Y(y)$ and $f_Y(y)$? for given

The Joint PDF $f_{XY}(x, y) = \begin{cases} 2 - x - y; & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0; & \text{Else where} \end{cases}$

Solution:

$$F_{XY}(x, y) = \int_{u=0}^{x} \int_{v=0}^{y} (2 - u - v)\, dv\, du$$

$$= \int_{u=0}^{x} \left[2v - uv - \frac{v^2}{2} \right]_0^y du$$

$$= \int_{u=0}^{x} \left[\left[2y - uy - \frac{y^2}{2} \right] - 0 \right] du$$

$$= \left[2yu - y\frac{u^2}{2} - \frac{y^2}{2} \cdot u \right]_0^x$$

$$= \left[2xy - \frac{x^2 y}{2} - \frac{x y^2}{2} \right]$$

$$\therefore F_{XY}(x, y) = 2xy - \frac{x^2 y}{2} - \frac{x y^2}{2}; \quad 0 \le x \le 1 \text{ and } 0 \le y \le 1$$

$$\therefore F_{XY}(x, y) = \begin{cases} 0; & -\infty \le x \le 0 \text{ and } -\infty \le y \le 0 \\ 2xy - \frac{x^2 y}{2} - \frac{x y^2}{2}; & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 1; & 1 \le x \le \infty \text{ and } 1 \le y \le \infty \end{cases}$$

2. Marginal PDF $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_{y=-\infty}^{y} f_{XY}(x, y)\, dy$$

$$= \int_{y=0}^{1} (2 - x - y)\, dy$$

$$= \left[2y - xy - \frac{y^2}{2} \right]_{y=0}^{1}$$

$$= \left[2 - x - \frac{1}{2} \right] - 0$$

$$= \frac{3}{2} - x$$

$$f_Y(y) = \int_{x=-\infty}^{x} f_{XY}(x, y)\, dx$$

$$= \int_{y=0}^{1} (2 - x - y)\, dy$$

$$= \left[2x - \frac{x^2}{2} - xy \right]_{x=0}^{1}$$

$$= \left[2 - \frac{1}{2} - y \right] - 0$$

$$= \frac{3}{2} - y$$

$$\therefore F_X(x) = \begin{cases} \frac{3}{2} - x; & 0 \le x \le 1 \\ 0; & \text{Else where} \end{cases}$$

$$\therefore F_Y(y) = \begin{cases} \frac{3}{2} - y; & 0 \le y \le 1 \\ 0; & \text{Else where} \end{cases}$$

3. Marginal CDF $F_X(x)$ and $F_Y(y)$

i. $F_X(x) = 0; \quad -\infty \le x \le 0$

ii.
$$F_X(x) = \int_{x=0}^{x} f_X(u)\, du$$
$$= \int_{x=0}^{x} \left[\frac{3}{2} - u\right] du$$
$$= \left[\frac{3}{2}u - \frac{u^2}{2}\right]_0^x$$
$$= \left[\frac{3}{2}x - \frac{x^2}{2}\right] - 0$$
$$= \frac{3}{2}x - \frac{x^2}{2}$$

iii. $F_X(x) = 0; \quad \le x \ge 0$

i. $F_Y(y) = 0; \quad -\infty \le y \le 0$

ii.
$$F_Y(y) = \int_{y=0}^{y} f_Y(v)\, dv$$
$$= \int_{x=0}^{y} \left[\frac{3}{2} - v\right] dv$$
$$= \left[\frac{3}{2}v - \frac{v^2}{2}\right]_0^y$$
$$= \left[\frac{3}{2}y - \frac{y^2}{2}\right] - 0$$
$$= \frac{3}{2}y - \frac{y^2}{2}$$

iii. $F_Y(y) = 0; \quad \le y \ge 0$

$$\therefore F_X(x) = \begin{cases} 0; & x \le 0 \\ \frac{3}{2}x - \frac{x^2}{2}; & 0 \le x \le 1 \\ 1; & x \ge 0 \end{cases}$$

$$\therefore F_Y(y) = \begin{cases} 0; & y \le 0 \\ \frac{3}{2}y - \frac{y^2}{2}; & 0 \le y \le 1 \\ 0; & y \ge 0 \end{cases}$$

Problem 3: Find the Joint PDF of two r.v X and Y, where CDF is given by

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-x^2})\,(1 - e^{-y^2}); & x \ge 0;\ y \ge 0 \\ 0; & y \ge 0 \end{cases}$$

Also find $P\{1 \le X \le 2, 1 \le Y \le 2\}$.

Solution:

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$
$$= \frac{\partial}{\partial x}(1 - e^{-x^2})\ \frac{\partial}{\partial y}(1 - e^{-y^2})$$
$$= \left[0 - e^{-x^2}(-2x)\right]\left[0 - e^{-y^2}(-2y)\right]$$
$$= 4xy\, e^{-x^2} e^{-y^2}$$

$$\therefore f_{XY}(x, y) = \begin{cases} 4xy\, e^{-x^2} e^{-y^2}; & x \ge 0,\ y \ge 0 \\ 0; & y \ge 0 \end{cases}$$

$P\{1 \le X \le 2, 1 \le Y \le 2\}$
$$= \int_{x=1}^{2} \int_{y=0}^{2} 4xy\, e^{-x^2} e^{-y^2}\, dy\, dx$$

$$= 4 \int_{x=1}^{2} x\, e^{-x^2} dx \int_{y=0}^{2} y\, e^{-y^2}\, dy \qquad \because \int uv = u \int v - \int du \int v$$

$$= 4 \left[\int_{x=1}^{2} d\left(\frac{e^{-x^2}}{-2} \right) \right] \left[\int_{y=1}^{2} d\left(\frac{e^{-y^2}}{-2} \right) \right] \qquad d\, e^{-x^2} = -2xe^{-x^2}\, dx$$

$$= \left[e^{-x^2} \right]_1^2 \left[e^{-y^2} \right]_1^2 \qquad \frac{d\, e^{-x^2}}{-2} = xe^{-x^2}\, dx$$

$$= \left(e^{-4} - e^{-1} \right)^2$$

$$= 0.12219$$

Problem 4. The Joint CDF of two random variables X and Y is given by

$$F_{XY}(x, y) = \begin{cases} c(2x + y); & 0 \le x \le 1;\ 0 \le y \le 2 \\ 0; & \text{Else where} \end{cases}$$

(i) Find the value of 'C'? (ii) Marginal CDF of 'X' and 'Y'.

Solution:

1. Total probability = 1

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx = 1$$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{2} c(2x + y) dy\, dx = 1$$

$$\Rightarrow \int_{x=0}^{1} c \left[2xy + \frac{y^2}{2} \right]_0^2 dx = 1$$

$$\Rightarrow \int_{x=0}^{1} c \left[4x + 2 - (0 + 0) \right] dx = 1$$

$$\Rightarrow c \left[4\frac{x^2}{2} + 2x \right]_0^1 = 1$$

$$\Rightarrow c \left(\frac{4}{2} + 2 \right) = 1$$

$$\Rightarrow c = \frac{1}{4}$$

2. Marginal PDF, CDF:

$$f_X(x) = \int_{y=-\infty}^{\infty} F_{XY}(x, y)\, dy$$

$$= \int_{y=0}^{2} c(2x + y)\, dy$$

$$= \frac{1}{4}\left[2xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{4}[4x + 2]$$

$$= x + \frac{1}{2}$$

$$\therefore f_X(x) = x + \frac{1}{2}$$

$F_X(x) = ?$

(a) $-\infty \le x \le 0; \quad F_X(x) = 0$

(b) $0 \le x \le 1; \quad F_X(x) = ?$

$F_X(x)$

$$= \int_{x=-\infty}^{0} f_X(x)\, dx + \int_{u=0}^{x} f_X(u)\, du$$

(first integral struck out, $=0$)

$$= \int_{u=0}^{x} \left(u + \frac{1}{2}\right) du$$

$$= \left[\frac{u^2}{2} + \frac{1}{2}u\right]_0^x$$

$$= \frac{1}{2}\left[x^2 + x\right]$$

(c) $0 \le x \le \infty; \quad F_X(x) = 0$

$$\therefore F_X(x) = \begin{cases} 0; & x \le 0 \\ \frac{x^2+x}{2}; & 0 \le x \le 1 \\ 1; & x \ge 1 \end{cases}$$

2. Marginal PDF, CDF:

$$f_Y(y) = \int_{x=-\infty}^{\infty} F_{XY}(x, y)\, dx$$

$$= \int_{x=0}^{1} c(2x + y)\, dx$$

$$= \frac{1}{4}\left[\frac{2x^2}{2} + xy\right]_0^1$$

$$= \frac{1}{4}[1 + y]$$

$$= \frac{y + 1}{4}$$

$$\therefore f_Y(y) = \frac{y + 1}{4}$$

$F_Y(y) = ?$

(a) $-\infty \le y \le 0; \quad F_Y(y) = 0$

(b) $0 \le y \le 1; \quad F_Y(y) = ?$

$F_Y(y)$

$$= \int_{x=-\infty}^{0} f_Y(y)\, dy + \int_{v=0}^{y} f_Y(v)\, dv$$

(first integral struck out, $=0$)

$$= \int_{v=0}^{y} \frac{v + 1}{4}\, dv$$

$$= \frac{1}{4}\left[\frac{v^2}{2} + v\right]_0^y$$

$$= \frac{1}{4}\left[\frac{y^2}{2} + y\right]$$

(c) $0 \le y \le \infty; \quad F_Y(y) = 0$

$$\therefore F_Y(y) = \begin{cases} 0; & x \le 0 \\ \frac{y^2+2y}{8}; & 0 \le x \le 2 \\ 1; & x \ge 2 \end{cases}$$

Problem 5. The Joint CDF of two random variables X and Y is given by

$$F_{XY}(x, y) = U(x)U(y)\left[1 - e^{\frac{-x^2}{2}} - e^{\frac{-y^2}{2}} + e^{\frac{-(x+y)^2}{2}}\right]$$

(i) Find $f_{XY}(x, y)$ (ii) $P\{0.5 \le X \le 1.5\}$

(iii) $P\{X \le 1 \cap Y \le 2\}$ (iv) $P\{-0.5 \le X \le 0.2, 1 \le y \le 2\}$

Solution: Given

$$F_{XY}(x, y) = \begin{cases} \left[1 - e^{\frac{-x^2}{2}} - e^{\frac{-y^2}{2}} + e^{\frac{-(x+y)^2}{2}}\right]; & x \ge 0,\ y \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

$$F_{XY}(x, y) = 1 - e^{\frac{-x^2}{2}} - e^{\frac{-y^2}{2}} + e^{\frac{-x^2}{2}} e^{\frac{-y^2}{2}}$$

$$= 1 - e^{\frac{-x^2}{2}} - e^{\frac{-y^2}{2}}\left(1 - e^{\frac{-y^2}{2}}\right)$$

$$F_{XY}(x, y) = \left[1 - e^{\frac{-x^2}{2}}\right]\left(1 - e^{\frac{-y^2}{2}}\right); \quad x \ge 0,\ y \ge 0$$

(a) $f_{XY}(x, y) = \dfrac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$

$$= \frac{\partial^2}{\partial x \partial y}\left(1 - e^{-\frac{x}{2}}\right)\left(1 - e^{-\frac{x}{2}}\right)$$

$$= \frac{\partial}{\partial x}\left(1 - e^{-\frac{x}{2}}\right) \frac{\partial}{\partial y}\left(1 - e^{-\frac{x}{2}}\right)$$

$$= \left(0 - e^{-\frac{x}{2}}\right)\left(\frac{-1}{2}\right) \cdot \left(0 - e^{-\frac{y}{2}}\right)\left(\frac{-1}{2}\right)$$

$$= \frac{1}{4} e^{-\frac{x}{2}} e^{-\frac{y}{2}}$$

$$= \frac{1}{4} e^{-\frac{(x+y)}{2}}$$

$$\boxed{\therefore f_{XY}(x, y) = \frac{1}{4} e^{-\frac{(x+y)}{2}}}$$

(b) $P(0.5 \le X \le 1.5) = \displaystyle\int_{x=0.5}^{1.5} \int_{y=0}^{\infty} \frac{1}{4} e^{-\frac{(x+y)}{2}}\, dy\, dx$

$$= \frac{1}{4} \int_{x=0.5}^{1.5} e^{-\frac{x}{2}} \left[\frac{e^{-\frac{y}{2}}}{-\frac{1}{2}}\right]_0^{\infty} dx$$

$$= -\frac{2}{4} \int_{x=0.5}^{1.5} e^{-\frac{x}{2}} \left(e^{-\infty} - 1\right) dx$$

$$= \frac{1}{2} \int_{x=0.5}^{1.5} e^{-\frac{x}{2}}\, dx$$

$$= \frac{1}{2} \left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_{0.5}^{1.5}$$

$$= -1\left[e^{-\frac{1.5}{2}} - e^{-\frac{0.5}{2}}\right]$$

$$= -[0.472 - 0.778]$$

$$= 0.306$$

$$\therefore P(0.5 \le X \le 1.5) = 0.306$$

$$(c)\ P(X \le 1, Y \le 2) = \int_{x=0}^{1}\int_{y=0}^{2} \frac{1}{4} e^{-\frac{(x+y)}{2}}\, dy\, dx$$

$$= \frac{1}{4}\int_{x=0}^{1} e^{-\frac{x}{2}} \left[\frac{e^{-\frac{y}{2}}}{-\frac{1}{2}}\right]_0^2 dx$$

$$= -\frac{2}{4}\int_{x=0}^{1} e^{-\frac{x}{2}} \left(e^{-2} - 1\right) dx$$

$$= -\frac{1}{2}\int_{x=0}^{1} e^{-\frac{x}{2}} [-0.632]\, dx$$

$$= 0.316 \left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_0^1$$

$$= -0.632\left[e^{-\frac{1}{2}} - 1\right]$$

$$= -[0.606 - 1]$$

$$= -0.632[-0.393]$$

$$= 0.248$$

$$\therefore P(X \le 1, Y \le 2) = 0.248$$

$$(d)\ P(0.5 \le X \le 2, 1 \le Y \le 3) = \int_{x=0.5}^{2}\int_{y=1}^{3} \frac{1}{4} e^{-\frac{(x+y)}{2}}\, dy\, dx$$

$$= \frac{1}{4}\int_{x=0.5}^{2} e^{-\frac{x}{2}} \left[\frac{e^{-\frac{y}{2}}}{-\frac{1}{2}}\right]_1^3 dx$$

$$= -\frac{1}{4}\int_{x=0.5}^{2} e^{-\frac{x}{2}} \left[e^{-\frac{3}{2}} - e^{-\frac{1}{2}}\right] dx$$

$$= 0.1915 \int_{x=0.5}^{2} e^{-\frac{x}{2}}\, dx$$

$$= 0.1915\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_{0.5}^{2}$$

$$= -0.383\left[e^{-1} - e^{-\frac{0.5}{2}}\right]$$

$$= 0.1577$$

$\therefore P(0.5 \le X \le 2, 1 \le Y \le 3) = 0.1577$

(e) $\therefore P(-0.5 \le X \le 0.2, 1 \le Y \le 3) = 0.036$

Problem 6. The Joint PDF of two random variables X and Y is given by

$$F_{XY}(x, y) = \begin{cases} a(2x + y^2); & 0 \le x \le 2, 2 \le y \le 3 \\ 0; & \text{otherwise} \end{cases}$$

(i) Find value of 'a'? (ii) $P\{X \le 1, Y > 3\}$

Ans: (i) $a = \frac{3}{136}$ (ii) $\frac{40}{136}$

Problem 7. The Joint PDF of two random variables X and Y is given by

$$F_{XY}(x, y) = \begin{cases} cxy\, e^{-(x^2+y^2)} & x \ge 0, y \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

(i) Find value of 'c'? (ii)Marginal distribution function of X and Y

(iii) Show that X and Y are independent (iv) $P\{X \le 1, Y \le 1\}$

Ans: (i) $c = 4$ (ii) $f_X(x) = 2x\, e^{-x^2}$, $F_X(x) = 1 - e^{-x^2}$

(iii) Independent. $f_X(x) f_Y(y) = f_{XY}(x, y)$ (iv) $P\{X \le 1, Y \le 1\} = 0.3995$

5.3 Statistical Independence

Two events A and B are said to be statistically independent if

$P(A \cap B) = P(AB) = P(A) \cdot P(B)$

Let event $A = \{X \le x\} = \{-\infty \le X \le x\}$, $B = \{Y \le y\} = \{-\infty \le Y \le y\}$. R.v X and Y are independent if,

$P\{-\infty \le X \le x \cap -\infty \le Y \le y\} = P\{-\infty \le X \le x\} \cdot P\{-\infty \le Y \le y\}$

$$\therefore F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$$

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Problem 8. Let $f_{XY}(x, y) = xe^{-x(1+y)}\, U(x)U(y)$. Check for independent?

Solution: The Condition is

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y) \tag{5.1}$$

$$f_X(x) = \int_{y=0}^{\infty} f_{XY}(x, y)\, dy$$

$$= \int_{y=0}^{\infty} xe^{-x(1+y)} U(x) U(y) dy$$

$$= xe^{-x} U(x)$$

$$f_Y(y) = \int_{x=0}^{\infty} f_{XY}(x, y)\, dx$$

$$= \int_{y=0}^{\infty} xe^{-x(1+y)} U(x) U(y) dy$$

$$= \frac{U(y)}{(1+y)^2}$$

From equation (5.1) $\Rightarrow xe^{-x(1+y)} U(x)U(y) \neq xe^{-x} U(x) \cdot \frac{U(y)}{(1+y)^2}$
So, X and Y are not independent.

Problem 9. $f_{XY}(x, y) = \frac{1}{12} e^{-\frac{x}{4}} e^{-\frac{y}{3}} U(x)U(y)$. Check independent X and Y.

Solution:

$$f_X(x) = \frac{1}{12} \int_{y=0}^{\infty} e^{-\frac{x}{4}} e^{-\frac{y}{3}} dy \qquad f_Y(y) = \frac{1}{12} \int_{x=0}^{\infty} e^{-\frac{x}{4}} e^{-\frac{y}{3}} dx$$

$$= \frac{e^{-\frac{x}{4}}}{12} \left[\frac{e^{-\frac{y}{3}}}{-\frac{1}{3}}\right]_0^{\infty} \qquad = \frac{e^{-\frac{y}{3}}}{12} \left[\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}}\right]_0^{\infty}$$

$$= \frac{e^{-\frac{x}{4}}}{12}(3) \qquad = \frac{e^{-\frac{y}{3}}}{12}(4)$$

$$= \frac{e^{-\frac{x}{4}}}{4} \qquad = \frac{e^{-\frac{x}{4}}}{3}$$

$\therefore f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$ So, X and Y are independent.

Problem 10. Check independent X and Y for given PDF.

$$f_{XY}(x, y) = \begin{cases} x + y; & 0 \le x \le 2, \text{ and } 0 \le y \le 1 \\ 0; & \text{otherwise} \end{cases}$$

Solution:

$$f_X(x) = \int_{y=0}^{1} (x+y)\,dy \qquad f_Y(y) = \int_{x=0}^{2} (x+y)\,dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2} \qquad = \left[\frac{x^2}{2} + xy\right]_0^2 = 2(1+y)$$

$f_X(x)f_Y(y) = (x + \frac{1}{2})(1 + y)2$

$\therefore f_{XY}(x, y) \neq f_X(x) \cdot f_Y(y)$ So, X and Y are not independent.

Problem 11. Check independent X and Y, find $P\{X \le 1, Y \le 1\}$ for given PDF.

$$f_{XY}(x, y) = \begin{cases} xye^{-\frac{x^2+y^2}{2}}; & x \geq 2, \text{ and } y \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

Solution:

$$f_X(x) = \int_{y=0}^{\infty} xy\, e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dy \qquad f_Y(y) = \int_{x=0}^{\infty} xy\, e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}\, dx$$

$$= xe^{-\frac{x^2}{2}} \left[y\, e^{-\frac{y^2}{2}}\right]_0^{\infty} \qquad = ye^{-\frac{y^2}{2}} \left[x\, e^{-\frac{x^2}{2}}\right]_0^{\infty}$$

let $\frac{y^2}{2} = t \Rightarrow 2y\,dy = dt$; let $\frac{x^2}{2} = t \Rightarrow 2x\,dx = dt$;

$y = 0 \Rightarrow t = 0; \quad y = \infty \Rightarrow t = \infty$ $\qquad x = 0 \Rightarrow t = 0; \quad x = \infty \Rightarrow t = \infty$

$$= xe^{-\frac{x^2}{2}} \int_{t=0}^{\infty} e^{-t} dt \qquad = ye^{-\frac{y^2}{2}} \int_{t=0}^{\infty} e^{-t} dt$$

$$= xe^{-\frac{x^2}{2}} \qquad = ye^{-\frac{y^2}{2}}$$

$\therefore f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$ So, X and Y are independent.

$$P\{X \le 1, Y \le 1\} = \int_0^1 \int_0^1 f_XY(x, y)\; dy\; dx$$

$$= \int_0^1 \int_0^1 xye^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

$$= \int_0^1 xe^{-\frac{x^2}{2}} \left[e^{-t}\right]_0^1$$

$$= (1 - e^{-1})(1 - e^{-1})$$

$\therefore P\{X \le 1, Y \le 1\} = (1 - e^{-1})^2$

5.4 Conditional Distribution and Density Functions

The conditional distribution function of a random variable 'X' given that same event 'B' is defined as

$$F_X(x/B) = P\{X \le x/B\} = \frac{P\{X \le x \cap B\}}{P\{B\}}; \qquad P(B) == 0$$

This is called conditional CDF for point conditioning.

Similarlly $f_X(x/B) = \frac{d}{dx} F_X(x/B)$ $\qquad \because P(B/A) = \frac{P(AB)}{P(B)}$

Let event 'B' is defined as $B = \{X \le b\}$. *i.e.,* $(-\infty \le X \le b)$. Now

$$F_X\left(x/(X \le b)\right) = P\left\{\frac{X \le x}{X \le b}\right\} = \frac{P\{(X \le x) \cap (X \le b)\}}{P\{X \le b\}}$$

This is called conditional CDF for interval conditioning.

$$F_X\left(x/(X \le b)\right) = \begin{cases} f_X(x); & x \ge b, \\ 0; & x \ge b \end{cases}$$

$$f_X\left(x/(X \le b)\right) = \begin{cases} \frac{d}{dx} F_X(x); & x \ge b \\ 0; & x \ge b \end{cases}$$

$$\int_{-\infty}^{b} f_X(x)dx = F_X\left(x/(x \le b)\right)$$

The above conditional r.v of single random variable 'X' can be extended into multiple random variable. i.e., two random variable X and Y.

Problem 12.

If $f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0, \\ 0; & \text{otherwise} \end{cases}$

1. Find $f_X(x), f_Y(y)$
2. Find $F_X(x), F_Y(y)$
3. $P(X < 1)$
4. $P(X < 1 \cap Y < 3)$
5. $P\{(X < 1)/(Y < 3)\}$
6. $P\{(X > 1 \cap Y < 2)/(X < 3)\}$
7. $P\{(X > 1 \cap Y < 2)/(Y > 3)\}$

Solution:

1. $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy$

$= \int_{y=0}^{\infty} e^{-(x+y)}\, dy$

$= e^{-x} \int_{y=0}^{\infty} e^{-y}\, dy$

$= e^{-x} \left[\dfrac{e^{-y}}{-1}\right]_0^{\infty}$

$= e^{-x}[0 + 1]$

$= e^{-x}$

$\therefore \quad f_X(x) = e^{-x}$

similarly $f_Y(y) = e^{-y}$

2. (i) $-\infty \leq X \leq 0; \quad F_X(x) = 0$

(ii) $0 \leq X \leq \infty;$

$F_X(x) = \int_{u=-\infty}^{x} f_x(u)\, du = \int_{u=0}^{x} e^{-u}\, du$

$= \left[\dfrac{e^{-u}}{-1}\right]_0^x = \left[\dfrac{e^{-x}}{-1}\right] - \dfrac{1}{-1}$

$= 1 - e^{-x}$

$\therefore F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0, \\ 0; & x \leq 0 \end{cases}$

$\therefore F_Y(y) = \begin{cases} 1 - e^{-y} & y \geq 0, \\ 0; & y \leq 0 \end{cases}$

3.

$P(X < x) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx$

$P(X < 1) = \int_{x=-\infty}^{1} \int_{x=-\infty}^{\infty} e^{-(x+y)}\, dy\, dx$

$= \int_{x=-\infty}^{1} \left[\dfrac{e^{-(x+y)}}{-1}\right]_0^{\infty} dy$

$= \int_{x=-\infty}^{1} e^{-x}\, dx$

$= \left[\dfrac{e^{-x}}{-1}\right]_0^1$

$= 1 - e^{-1}$

Other method:

$P(X < x) = \int_{x=-\infty}^{\infty} f_X(x)\, dx$

$P(X < 1) = \int_{x=-\infty}^{1} e^{-x}\, dx$

$= \left[\dfrac{e^{-x}}{-1}\right]_0^1$

$= 1 - e^{-1}$

4. $$P(X < 1 \cap Y < 3) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{3} e^{-(x+y)}\, dy\, dx$$

$$= \left(1 - e^{-3}\right)\left(1 - e^{-1}\right)$$

5. $$P(Y < 3) = \int_{y=0}^{3} f_Y(y)\, dy$$

$$= \int_{y=0}^{3} e^{-x}\, dy$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^3$$

$$= 1 - e^{-3}$$

$$P\left(\frac{X \leq x}{X \leq b}\right) = \frac{P\{(X \leq x) \cap (X \leq b)\}}{P\{X \leq b\}}$$

$$P\left(\frac{X \leq 1}{Y \leq 3}\right) = \frac{P\{(X \leq 1) \cap (Y \leq 3)\}}{P\{Y \leq 3\}}$$

$$= \frac{(1 - e^{-3})(1 - e^{-1})}{(1 - e^{-3})}$$

$$= 1 - e^{-1}$$

(6) $$P\left(\frac{(X \geq 1) \cap (Y < 2)}{X \leq 3}\right) = \frac{e^{-1} - e^{-3}}{1 - e^{-3}}$$

(7) $$P\left(\frac{(X \geq 1) \cap (Y < 2)}{X \leq 3}\right) = e^{2} - 1$$

5.5 Discrete random variable

1.

$$F_{XY}(x,y) = P\{X \le x_n \cap Y \le y_m\} = P(X = x_n, Y = y_m)$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{XY}(X = x_n, Y = y_m)\, U(x - x_n)\, U(y - y_m)$$

2.

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{XY}(x_n, y_m)\, \delta(x - x_n)\, \delta(y - y_m)$$

3.

$$F_X(x) = F_{XY}(x, \infty) = P\{X \le x_n \cap Y \le \infty\}$$
$$= \sum_{n=-\infty}^{\infty} f_{XY}(x_n, y_m)\, U(x - x_n)\, U(y - \infty)$$
$$\therefore F_X(x) = \sum_{n=-\infty}^{\infty} f_{XY}(x,y)\, U(x - x_n)$$

4.

$$F_Y(y) = F_{XY}(\infty, y) = \sum_{m=-\infty}^{\infty} f_{XY}(x,y)\, U(y - y_m)$$

5.

$$f_X(x) = \sum_{m=-\infty}^{y_m} F_{XY}(x,y)\, U(y - y_m) = \frac{d}{dx} F_X(x)$$

6.

$$f_Y(y) = \sum_{n=-\infty}^{x_n} F_{XY}(x,y)\, U(x - x_n) = \frac{d}{dx} F_X(x)$$

Problem 13. The joint space for two random variable X and Y, and corresponding probabilities are shown in table.

(x, y)	(1,1)	(2,1)	(3,3)
$P(x, y)$	0.2	0.3	0.5

Find and plot:

1. $F_{XY}(x, y)$ and $f_{XY}(x, y)$
2. $F_X(x)$ and $f_X(x)$
3. $F_Y(y)$ and $f_Y(y)$
4. Find $P\{0 \le X \le 1 \cap 0 \le Y \le 3\}$
5. Find $P\{0 \le X \le 2 \cap 0 \le Y \le 2\}$
6. Find $P\{0 \le X \le 2 \cap 0 \le Y < 2\}$

Solution:

$$F_{XY}(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f_{XY}(X = x_n, Y = y_m)\, U(x - x_n)\, U(y - y_m)$$

$$F_{XY}(x,y) = \sum_{n=1}^{3}\sum_{m=1}^{3} f_{XY}(X = x_n, Y = y_m)\, U(x - x_n)\, U(y - y_m)$$

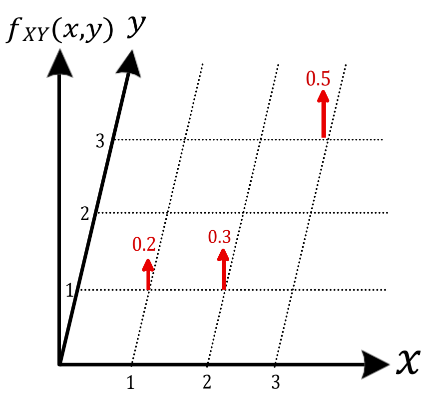
$$= P(1,1)\, U(x-1)\, U(y-1) + P(2,1)\, U(x-2)\, U(y-1)$$

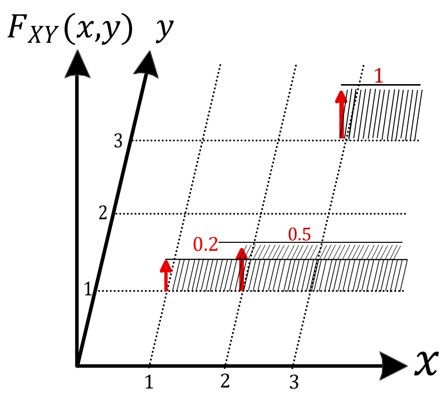
$$+ P(3,3)\, U(x-3)\, U(y-3)$$

$$F_{XY}(x,y) = 0.2\, U(x-1)\, U(y-1) + 0.3\, U(x-2)\, U(y-1) + 0.5\, U(x-3)\, U(y-3)$$

$$\therefore F_{XY}(x,y) = 0.2\, U(x-1)\, U(y-1) + 0.3\, U(x-2)\, U(y-1) + 0.5\, U(x-3)\, U(y-3)$$

$$\therefore f_{XY}(x,y) = 0.2\, \delta(x-1)\, \delta(y-1) + 0.3\, \delta(x-2)\, \delta(y-1) + 0.5\, \delta(x-3)\, \delta(y-3)$$





2. Marginal PDF and CDF: $f_X(x)$ and $F_X(x)$

$$F_{XY}(x, \infty) = F_X(x)$$

$$= 0.2\, U(x-1)\, U(\infty - 1) + 0.3\, U(x-2)\, U(\infty - 1) + 0.5\, U(x-3)\, U(\infty - 3)$$

$$= 0.2\, U(x-1) + 0.3\, U(x-2) + 0.5\, U(x-3)$$

$$\therefore F_{XY}(x) = 0.2\, U(x-1) + 0.3\, U(x-2) + 0.5\, U(x-3)$$

$$\therefore f_{XY}(x) = \frac{d}{dx} F_X(x) = 0.2\, \delta(x-1) + 0.3\, \delta(x-2) + 0.5\, \delta(x-3)$$

3. Marginal PDF and CDF: $f_Y(y)$ and $F_Y(y)$

$$F_{XY}(\infty, y) = F_Y(y)$$

$$= 0.2\, U(\infty - 1)\, U(y-1) + 0.3\, U(\infty - 2)\, U(y-1) + 0.5\, U(\infty - 3)\, U(y-3)$$

$$= 0.2\, U(y-1) + 0.3\, U(y-1) + 0.5\, U(y-3)$$

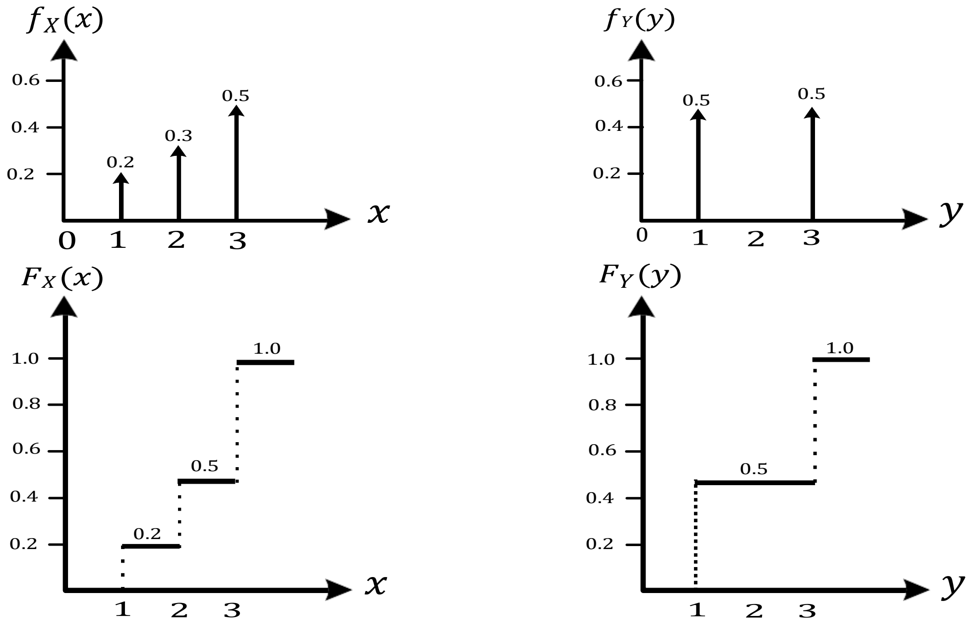
$$\therefore F_{XY}(y) = 0.5\, U(y-1) + 0.5\, U(y-3)$$

$$\therefore f_{XY}(y) = \frac{d}{dy} F_Y(y) = 0.5\, \delta(y-1) + 0.5\, \delta(y-3)$$

4. $P\{0 \leq X \leq 1 \cap 0 \leq Y \leq 3\} = P(1, 1) = 0.2$

5. $P\{0 \leq X \leq 2 \cap 0 \leq Y \leq 2\} = P(1, 1) + P(2, 1) = 0.5$

6. $P\{0 \leq X \leq 2 \cap 0 \leq Y < 2\} = P(1, 1) = 0.2$



Problem 14. The joint space for two random variable X and Y, and corresponding probabilities are shown in table.

(x, y)	(1,1)	(2,2)	(3,3)	(4,4)
$P(x_n, y_n)$	0.05	0.35	0.45	0.15

Find and plot:

1. $F_{XY}(x, y)$ and $f_{XY}(x, y)$
2. $F_X(x)$ and $f_X(x)$
3. $F_Y(y)$ and $f_Y(y)$
4. Find $P\{0.5 \leq X \leq 1.5\}$
5. Find $P\{X \leq 2 \cap Y \leq 2\}$
6. Find $P\{1 < X \leq 2, Y \leq 2\}$

Solution:

$$F_{XY}(x, y) = \sum_{n=1}^{4} \sum_{m=1}^{4} f_{XY}(X = x_n, Y = y_m)\, U(x - x_n)\, U(y - y_m)$$

$$= P(1, 1)\, U(x - 1)\, U(y - 1) + P(2, 2)\, U(x - 2)\, U(y - 1)$$

$$+ P(3, 3)\, U(x - 3)\, U(y - 3) + P(4, 4)\, U(x - 4)\, U(y - 4)$$

$$F_{XY}(x, y) = 0.05\, U(x - 1)\, U(y - 1) + 0.35\, U(x - 2)\, U(y - 2)$$

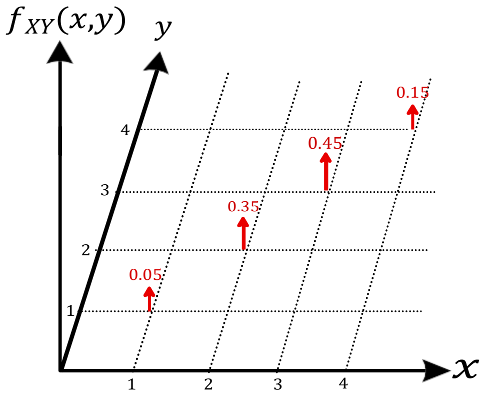
$$+ 0.45\, U(x - 3)\, U(y - 3) + 0.15\, U(x - 4)\, U(y - 4)$$

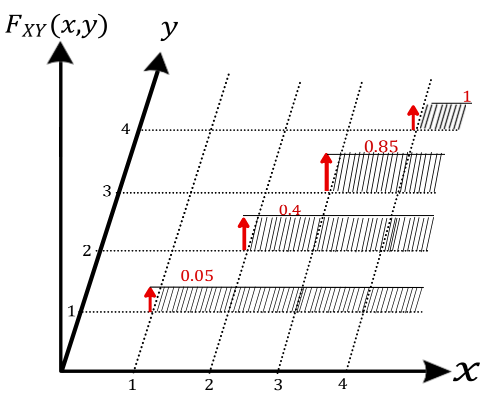
$$\therefore F_{XY}(x, y) = 0.05\, U(x - 1)\, U(y - 1) + 0.35\, U(x - 2)\, U(y - 2)$$

$$+ 0.45\, U(x - 3)\, U(y - 3) + 0.15\, U(x - 4)\, U(y - 4)$$

$$\therefore f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) = 0.05\ \delta(x-1)\ \delta(y-1) + 0.35\ \delta(x-2)\ \delta(y-2)$$

$$+ 0.45\ \delta(x-3)\ \delta(y-3) + 0.15\ \delta(x-4)\ U(y-4)$$





2. Marginal PDF and CDF: $f_X(x)$ and $F_X(x)$

$F_{XY}(x, \infty) = F_X(x)$

$= 0.05\ U(x-1)\ U(\infty-1) + 0.35\ U(x-2)\ U(\infty-2)$

$\quad + 0.45\ U(x-3)\ U(\infty-3) + 0.15\ U(x-4)\ U(\infty-4)$

$= 0.05\ U(x-1) + 0.35\ U(x-2) + 0.45\ U(x-3) + 0.15\ U(x-4)$

$\therefore F_X(x) = 0.05\ U(x-1) + 0.35\ U(x-2) + 0.45\ U(x-3) + 0.15\ U(x-4)$

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = 0.05\ \delta(x-1) + 0.35\ \delta(x-2)$$

$$+ +0.45\ \delta(x-3) + 0.15\ \delta(x-4)$$

3. Marginal PDF and CDF: $f_Y(y)$ and $F_Y(y)$

$F_{XY}(\infty, y) = F_Y(y)$

$= 0.05\ U(\infty-1)\ U(y-1) + 0.35\ U(\infty-2)\ U(y-2)$

$\quad + 0.45\ U(\infty-3)\ U(y-3) + 0.15\ U(\infty-4)\ U(y-4)$

$= 0.05\ U(y-1) + 0.35\ U(y-2) + 0.45\ U(y-3) + 0.15\ U(y-4)$

$\therefore F_Y(y) = 0.05\ U(y-1) + 0.35\ U(y-2) + 0.45\ U(y-3) + 0.15\ U(y-4)$

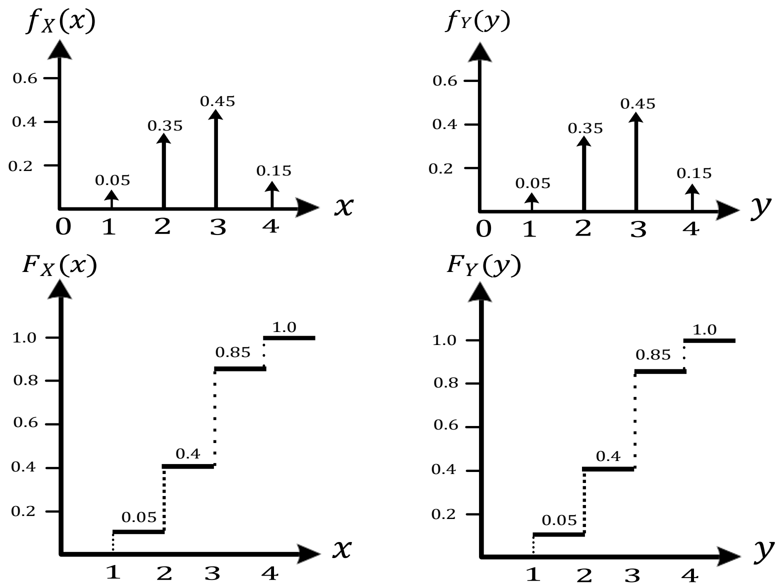
$$\therefore f_Y(y) = \frac{d}{dx} F_X(x) = 0.05\ \delta(y-1) + 0.35\ \delta(y-2)$$

$$+ 0.45\ \delta(y-3) + 0.15\ \delta(y-4)$$

4. $P\{0.5 \le X \le 1.5\} = P(1, 1) = 0.05$

5. $P\{X \le 2 \cap Y \le 2\} = P(2, 1) + P(2, 2) + P(2, 3) = 0 + 0.35 + 0 = 0.35$

6. $P\{1 < X \le 2, Y \le 2\} = P(1,1) + P(1,2) = 0.05 + 0 = 0.05$



Problem 15. The joint space for two random variable X and Y, and corresponding probabilities are shown in table.

$\frac{X}{Y}$	1	2	3
1	0.2	0.1	0.2
2	0.15	0.2	0.15

Find and plot:

1. Joint and marginal distribution function
2. Joint and marginal density function

Solution: Given data is

$P(1,1) = 0.2,\quad P(2,1) = 0.1,\quad P(3,1) = 0.2,$

$P(1,2) = 0.15,\quad P(2,2) = 0.2,\quad P(3,2) = 0.15$

$$F_{XY}(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} P(x_n, y_n)\, U(x - x_n)\, U(y - y_m)$$

$$F_{XY}(x,y) = \sum_{n=1}^{3}\sum_{m=1}^{3} P(x_n, y_n)\, U(x - x_n)\, U(y - y_m)$$

$$= P(1,1)\, U(x-1)\, U(y-1) + P(1,2)\, U(x-1)\, U(y-2)$$
$$+ P(2,1)\, U(x-2)\, U(y-1) + P(2,2)\, U(x-2)\, U(y-2)$$
$$+ P(3,1)\, U(x-3)\, U(y-1) + P(3,2)\, U(x-3)\, U(y-2)$$

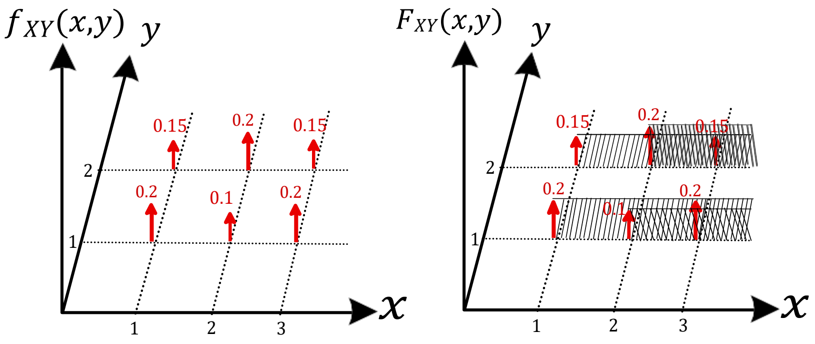
$$F_{XY}(x,y) = 0.2\, U(x-1)\, U(y-1) + 0.15\, U(x-1)\, U(y-2)$$
$$+ 0.1\, U(x-2)\, U(y-1) + 0.2\, U(x-2)\, U(y-2)$$
$$+ 0.2\, U(x-3)\, U(y-1) + 0.15\, U(x-3)\, U(y-2)$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_X XY(x, y)$$

$$= 0.2\ \delta(x-1)\ \delta(y-1) + 0.15\ \delta(x-1)\ \delta(y-2)$$

$$+ 0.1\ \delta(x-2)\ \delta(y-1) + 0.2\ \delta(x-2)\ \delta(y-2)$$

$$+ 0.2\ \delta(x-3)\ \delta(y-1) + 0.15\ \delta(x-3)\ \delta(y-2)$$



2. Marginal Distribution PDF and CDF: $f_X(x)$ and $F_X(x)$

$$F_X(x) = F_{XY}(x, \infty)$$

$$= 0.2\ U(x-1)\ U(\infty-1) + 0.15\ U(x-1)\ U(\infty-2)$$

$$+ 0.1\ U(x-2)\ U(\infty-1) + 0.2\ U(x-2)\ U(\infty-2)$$

$$+ 0.2\ U(x-3)\ U(\infty-1) + 0.15\ U(x-3)\ U(\infty-2)$$

$$F_X(x) = 0.2\ U(x-1) + 0.15\ U(x-1)$$

$$+ 0.1\ U(x-2) + 0.2\ U(x-2))$$

$$+ 0.2\ U(x-3) + 0.15\ U(x-3)$$

$$= 0.35\ U(x-1) + 0.3\ U(x-2) + 0.35\ U(x-3)$$

$$\therefore\ F_X(x) = 0.35\ U(x-1) + 0.3\ U(x-2) + 0.35\ U(x-3)$$

$$\therefore\ f_X(x) = \frac{d}{dx} F_X(x) = 0.35\ \delta(x-1) + 0.3\ \delta(x-2) + 0.35\ \delta(x-3)$$

3. Marginal Distribution PDF and CDF: $f_Y(y)$ and $F_Y(y)$

The marginal distribution function of r.v 'Y' is $F_Y(y)$ and it is obtained by subtituting $x = \infty$ in $F_{XY}(x, y)$.

$$F_Y(y) = F_{XY}(y, \infty)$$

$$= 0.2\ U(\infty-1)\ U(y-1) + 0.15\ U(\infty-1)\ U(y-2)$$

$$+ 0.1\ U(\infty-2)\ U(y-1) + 0.2\ U(\infty-2)\ U(y-2)$$

$$+ 0.2\ U(\infty-3)\ U(y-1) + 0.15\ U(\infty-3)\ U(y-2)$$

$$F_Y(y) = 0.2\ U(y-1) + 0.15\ U(y-2)$$

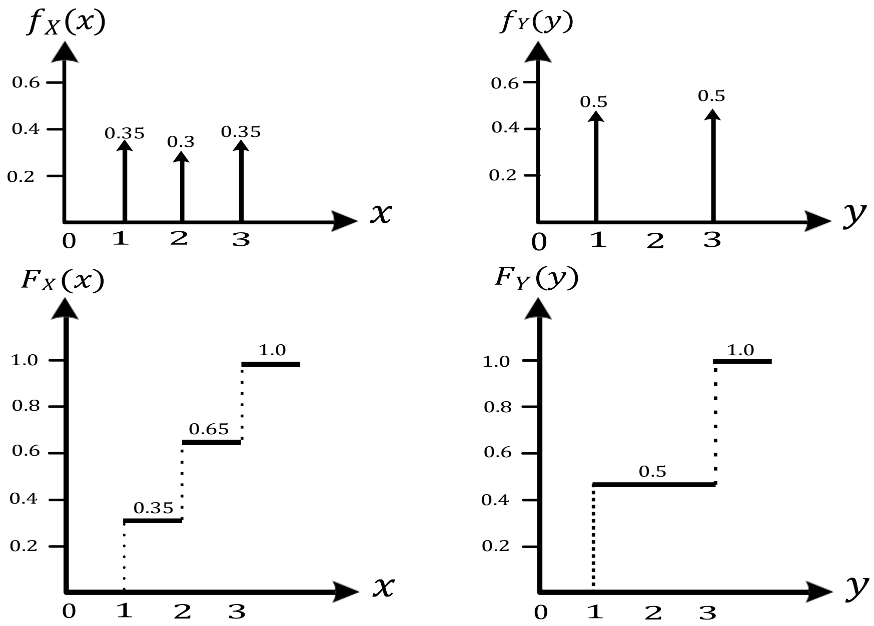
$$+ 0.1\ U(y-1) + 0.2\ U(y-2))$$

$$+ 0.2\ U(y-1) + 0.15\ U(y-2)$$

$$= 0.5\ U(y-1) + 0.5\ U(y-2)$$

$$\therefore\ F_Y(y) = 0.5\ U(y-1) + 0.5\ U(y-2)$$

$$\therefore\ f_Y(y) = \frac{d}{dy} F_Y(y) = 0.5\ \delta(y-1) + 0.5\ \delta(y-2)$$



Problem 16. Discrete r.v X and Y have a joint distribution function

$$F_{XY}(x, y) = 0.1\ U(x+4)\ U(y-1) + 0.15\ U(x+3)\ U(y+5)$$
$$+ 0.17\ U(x+1)\ U(y-3) + 0.05\ U(x)\ U(y-1)$$
$$+ 0.18\ U(x-2)\ U(y+2) + 0.23\ U(x-3)\ U(y-4)$$
$$+ 0.12\ U(x-4)\ U(y+3)$$

1. Sketch and plot $F_{XY}(x, y)$ $f_{XY}(x, y)$
2. Find and plot Marginal PDF and CDF
3. $P\{-1 < X \leq 4 \cap -3 < Y \leq 3\}$
4. $P\{X < 1, Y \leq 2\}$

5.6 Conditional Distribution and density for discrete r.v

Let X and Y are discrete random variable with values x_i, $i = 1, 2, 3, \ldots, N$ and y_j, $j = 1, 2, 3, \ldots, M$ respectively and probabilities are $P(X_i)$ and $P(Y_j)$ respectively. The probability of joint occurance of x_i and y_j is denoted by $P(x_i, y_j)$.

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\delta(x - x_i); \quad f_Y(y) = \sum_{j=1}^{M} P(y_i)\delta(y - y_j)$$

$$f_{XY}(x, y) = \sum_{i=1}^{N}\sum_{j=1}^{M} P(x_i, y_j)\delta(x - x_i)\,\delta(y - y_j)$$

Conditional distribution function,

$$F_X(x|y = y_k) = \frac{\sum_{i=1}^{N} P(x_i, y_k)U(x - x_i)}{p(y_k)}$$

$$f_X(x|y = y_k) = \frac{\sum_{i=1}^{N} P(x_i, y_k)\delta(x - x_i)}{p(y_k)}$$

$$F_Y(y|x = x_k) = \frac{\sum_{j=1}^{M} P(x_k, y_j)U(y - y_j)}{p(x_k)}$$

$$f_Y(y|x = x_k) = \frac{\sum_{j=1}^{M} P(x_k, y_j)\delta(y - y_j)}{p(x_k)}$$

Problem 17. Let $P(x_1, y_1) = \frac{2}{15}$, $P(x_2, y_1) = \frac{3}{15}$, $P(x_2, y_2) = \frac{1}{15}$, $P(x_1, y_3) = \frac{4}{15}$, $P(x_2, y_3) = \frac{5}{15}$. Find $f_X(x|Y = y_3)$?

$$f_X(x|y = y_3) = \frac{\sum_{i=1}^{3} P(x_i, y_3)\delta(x - x_i)}{p(y_3)}$$

$$P(y_3) = P(x_1, y_3) + P(x_2, y_3)$$

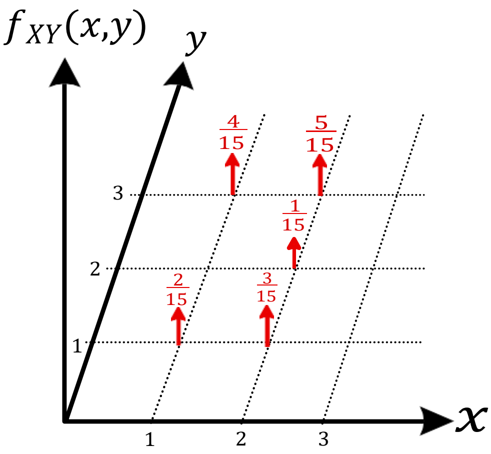
$$= \frac{4}{15} + \frac{5}{15} = \frac{9}{15} = \frac{3}{5}$$

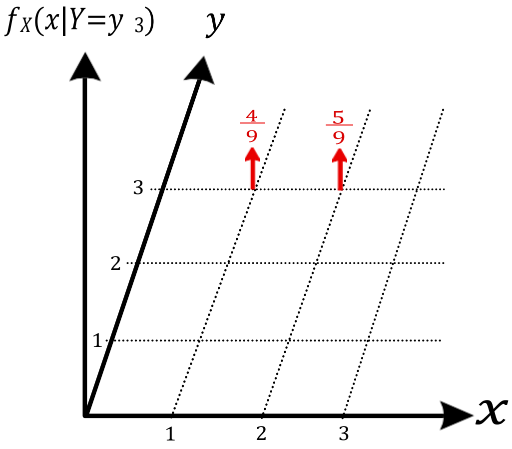
$$f_X(x|y = y_3) = \frac{P(x_1, y_3)\delta(x - x_1) + P(x_2, y_3)\delta(x - x_2) + P(x_3, y_3)\delta(x - x_3)}{P(y_3)}$$

$$= \frac{\frac{4}{15}\delta(x - x_1) + \frac{5}{15}\delta(x - x_2) + 0 \cdot \delta(x - x_3)}{\frac{3}{5}}$$

$$= \frac{5}{3} \times \frac{4}{15}\delta(x - x_1) + \frac{5}{3} \times \frac{5}{15}\delta(x - x_2)$$

$$f_X(x|y = y_3) = \frac{4}{9}\delta(x - x_1) + \frac{5}{9}\delta(x - x_2)$$





Problem 18. The following table represents the Joint distribution of the discrete r.v X and Y.

X / Y	1	2	3
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	0	$\frac{1}{9}$	$\frac{1}{5}$
3	$\frac{1}{18}$	$\frac{1}{4}$	$\frac{2}{15}$

1. Find $F_X(x|y=2)$
2. Find $F_Y(y|x=3)$

Find and plot:

3. Find $P\{X \le 2, Y=3\}$
4. $P\{Y \le 2\}$
5. $P\{X+Y < 4\}$
6. $\dfrac{P\{X \le 2, Y < 3\}}{P\{X < 3\}}$

Solution:

1. $F_X(x|y=2)$

$$= \frac{P\{X < x \cap y = 2\}}{P\{y=2\}}$$

$$= \frac{\sum_{i=1}^{3} P(x_i, y=2)U(x-x_i)}{p(y=2)}$$

$$= \frac{P(1,2)U(x-1) + P(2,2)U(x-2) + P(3,2)U(x-3)}{P(1,2) + P(2,2) + P(3,2)}$$

$$= \frac{0 + \frac{1}{9}U(x-2) + \frac{1}{5}U(x-3)}{\frac{1}{9} + \frac{1}{5}}$$

$$= \frac{5}{14}U(x-2) + \frac{9}{14}U(x-3)$$

2. $F_X(y|x=3)$

$$= \frac{P\{X = 3 \cap Y < y_j\}}{P\{x = 3\}} = \frac{\sum_{j=1}^{3} P(x = 3, y_j)U(y - y_j)}{p(x = 3)}$$

$$= \frac{P(3,1)U(y-1) + P(3,2)U(y-2) + P(3,3)U(y-3)}{P(3,1) + P(3,2) + P(3,3)}$$

$$= \frac{0 + \frac{1}{5}U(y-2) + \frac{2}{15}U(y-3)}{\frac{1}{9} + \frac{2}{15}}$$

$$= \frac{3}{5}U(y-2) + \frac{2}{5}U(y-3)$$

3. $P\{X \leq 2, Y = 3\} = P(1,3) + P(2,3) = \frac{1}{18} + \frac{1}{4} = \frac{11}{18}$

4. $P\{X + Y < 4\} = P(1,1) + P(1,2) + P(2,1) = \frac{1}{12} + 0 + \frac{1}{6} = \frac{1}{4}$

5.

$$P\{Y \leq 2\} = P(1,1) + P(1,2) + P(2,1) + P(2,2) + P(3,1) + P(3,2)$$

$$= \frac{1}{12} + 0 + \frac{1}{6} + \frac{1}{9} + 0 + \frac{1}{5}$$

$$=$$

6.

$$\frac{P\{X \leq 2 \cap Y < 3\}}{P\{X < 3\}} = \frac{P(1,1) + P(1,2) + P(2,1) + P(2,2)}{P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2) + P(2,3)}$$

$$= \frac{\frac{1}{12} + \frac{1}{6} + 0 + \frac{1}{9}}{\frac{1}{12} + 0 + \frac{1}{18} + \frac{1}{6} + \frac{1}{9} + \frac{1}{4}}$$

$$= \frac{(\frac{13}{36})}{(\frac{24}{36})} = \frac{13}{24}$$

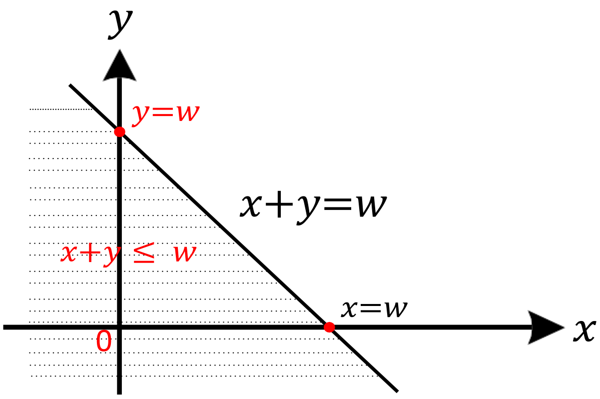
5.7 Sum of two independent random variables

Let 'W' be a random variable equal to the sum of two independent random variables X and Y.

$$W = X + Y$$

The probability of $W \leq w$ can be written as

$$f_W(w) = P\{W \leq w\} = P\{X + Y \leq w\} = P\{-\infty \leq W \leq w\} = \int_{u=-\infty}^{w} f_W(u)du$$



$$f_W(w) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_X(x)\, f_Y(y)\, dy\, dx$$

$$= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_X(x)\, f_Y(y)\, dx\, dy \qquad \because \text{ Independent}$$

$$\Rightarrow \int_{x=-\infty}^{w-y}\int_{y=-\infty}^{\infty} f_X(x)\, f_Y(y)dy\, dx \quad \text{(or)} \quad = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{w-x} f_X(x)\, f_Y(y)dy\, dx$$

Take integral and differentiate wrt to 'x'

$$\int_{x=-\infty}^{w-y} f_X(x)\, dx = \Big[f_X(x) \Big]_{x=-\infty}^{w-y}$$

$$= f_X(w-y) - \underbrace{f_X(-\infty)}_{=0}$$

$$= f_X(w-y)$$

Take integral and differentiate wrt to 'y'

$$\int_{y=-\infty}^{w-x} f_Y(y)\, dy = \Big[f_Y(y) \Big]_{y=-\infty}^{w-x}$$

$$= f_Y(w-x) - \underbrace{f_Y(-\infty)}_{=0}$$

$$= f_Y(w-x)$$

$$\therefore f_W(w) = \int_{y=-\infty}^{\infty} f_Y(y) f_X(w-y)dy \quad \text{(or)} \quad = \int_{x=-\infty}^{\infty} f_X(x) f_Y(w-x)dx$$

$$= f_X(x) \circledast f_Y(y) \qquad\qquad = f_X(x) \circledast f_Y(y)$$

$\therefore$ $\boxed{f_W(w) = f_X(x) * f_Y(y)}$

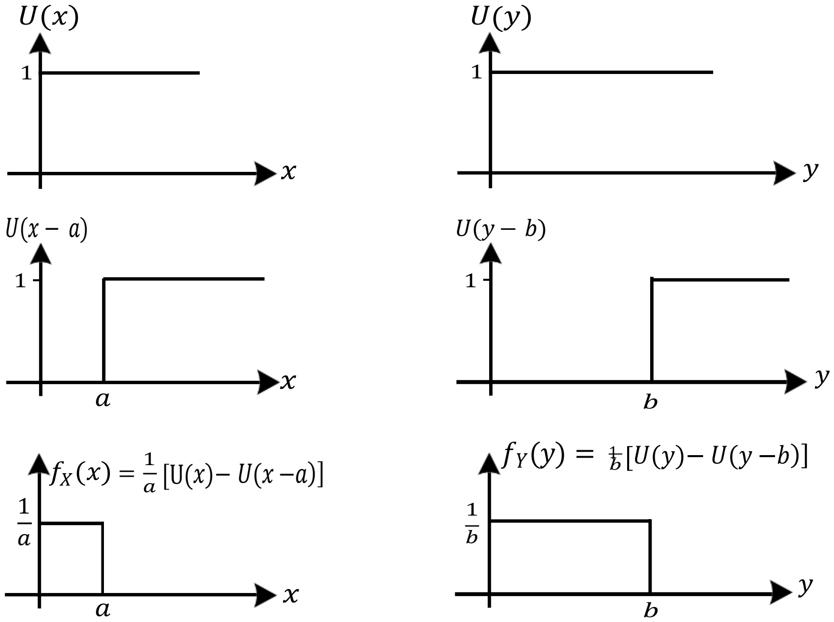
This expression is recognised as a convolutional integral.

$\therefore$ The density function of the sum of two statistically independent random variable is convolution of their individual density function.

Problem 19: Find the sum of two independent r.v $W = X + Y$, whose PDF are

$$f_X(x) = \frac{1}{a}\left[U(x) - U(x-a)\right]; \qquad x \geq 0$$

$$f_Y(y) = \frac{1}{b}\left[U(y) - U(y-b)\right]; \qquad y \geq 0, \text{ where } 0 < a < b$$



We know that, the density function of sum of two independent r.v is the convolution of their individual density function i.e.,

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

Here. range z is 0 to $a + b$ and $0 < a < b$

Case (i): at $z = 0$

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(-x)\, dx$$

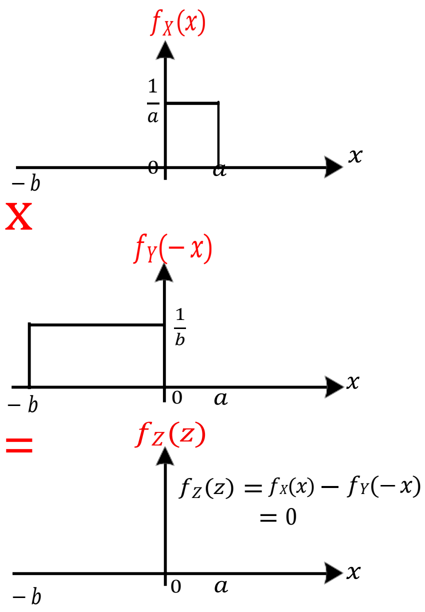
$$= 0$$

Case (ii): at $z = a$

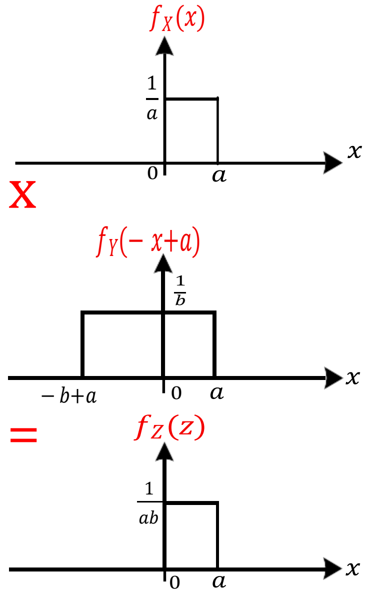
$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(a-x)\, dx$$

$$= \int_{x=-\infty}^{\infty} \frac{1}{ab}\, dx = \int_0^a \frac{1}{ab}\, dx$$

$$= \frac{1}{ab}\left[x\right]_0^a = \frac{a}{ab} = \frac{1}{b}$$



(i)



(ii)

Case (iii): at $z = b$

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(b-x)\, dx$$

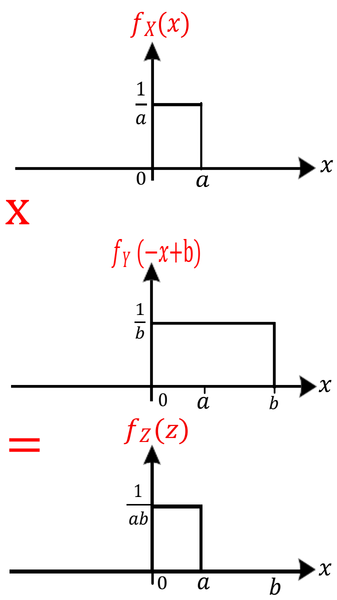
$$= \int_0^a \frac{1}{ab}\, dx$$

$$= \frac{1}{ab}\, x \Big|_0^a = \frac{a}{ab} = \frac{1}{b}$$

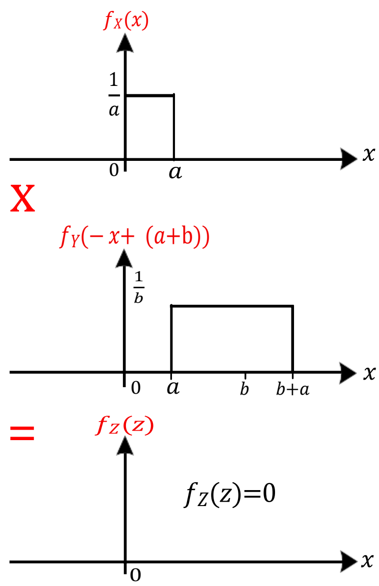
Case (iv): at $z = a + b$

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y((a+b)-x)\, dx$$

$$= 0$$

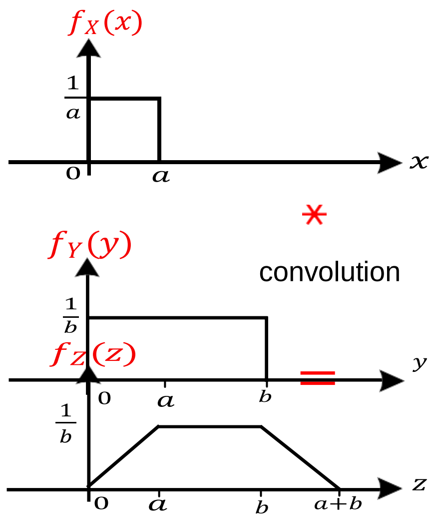


(iii)



(iv)

$$\therefore f_X(z) = \begin{cases} \frac{z}{ab}; & 0 \le w \le a \\ \frac{1}{b}; & a \le w \le b \\ \frac{a+b-z}{ab}; & b \le w \le a+b \\ 0; & w \ge a \\ 0; & -\infty \le w \le 0 \end{cases}$$



(v)

Problem 20: Two independent r.v X and Y have PDF is

$$f_X(x) = \begin{cases} xe^{-x}\, U(x); & x \ge 0 \\ 0; & \text{other wise} \end{cases} \qquad f_Y(y) = \begin{cases} 1; & 0 \le y \ge 1 \\ 0; & \text{other wise} \end{cases}$$

Calculate the $f_Z(z)$ when $Z = X + Y$.

Solution: $f_Y(y)$ can be written as $f_Y(y) = U(y) - U(y-1)$

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

$$= \int_{x=-\infty}^{\infty} xe^{-x}\, U(x) \Big[U(z-x) - U(z-x-1) \Big]\, dx$$

$$= \underbrace{\int_{x=-\infty}^{\infty} xe^{-x}\, U(x)\, U(z-x)\, dx}_{①} - \underbrace{\int_{x=-\infty}^{\infty} xe^{-x}\, U(x)\, U(z-x-1)\, dx}_{②}$$

consider integral ①

$$\int_{x=-\infty}^{\infty} xe^{-x}\, U(x)\, U(z-x)\, dx$$

$$= \int_{x=0}^{z} xe^{-x}\, dx$$

$$= \Big[x \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} \Big]_0^z \qquad \because \int uv = u \int v - \int du \int v$$

$$= \Big[x \cdot \frac{e^{-x}}{-1} - \frac{e^{-x}}{1} \Big]_0^z$$

$$= \left[z \cdot \frac{e^{-z}}{-1} - \frac{e^{-z}}{1} \right] - \left[0 - 1 \right]$$

$$= 1 - e^{-z}(1 + z)$$

consider integral ②

$$\int_{x=-\infty}^{\infty} x e^{-x}\, U(x)\, U(z - x - 1)\, dx$$

$$= \int_{x=0}^{z-1} x e^{-x}\, dx$$

$$= \left[x \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} \right]_0^{z-1} \qquad \because \int uv = u \int v - \int du \int v$$

$$= \left[x \cdot \frac{e^{-x}}{-1} - \frac{e^{-x}}{1} \right]_0^{z-1}$$

$$= \left[(z - 1) \cdot \frac{e^{-(z-1)}}{-1} - \frac{e^{-(z-1)}}{1} \right] - \left[0 - 1 \right]$$

$$= 1 - ze^{-(z-1)} + e^{-(z-1)} - e^{-(z-1)}$$

$$= 1 - ze^{-(z-1)}$$

$\therefore f_Z(z) =$ ① − ②

$$= 1 - e^{-z}(1 + z) - \{1 - ze^{-(z-1)}\}$$

$$= 1 - e^{-z}(1 + z) - 1 + ze^{-(z-1)}$$

$$= -e^{-z}(1 + z) + ze^{-z} \cdot e^{1}$$

$$\therefore f_Z(z) = e^{-z}\left[-1 - z + ze \right]$$

Problem 21: If X and Y are two r.v which are gaussian, if a r.v 'Z' is defined as $W = X + Y$. Find $f_W(w)$.

We know that Gaussian r.v density function

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \text{ Normalized Gaussion r.v}$$

Let X and Y be two normalized Gaussian r.v $\sigma_X^2 = \sigma_Y^2 = 1, \quad m_X = m_y = 0$ then

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$f_W(w) = f_X(x) * f_Y(y) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx$$

$$= \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{(w-x)^2}{2}}\, dx$$

$$= \frac{1}{2\pi}\int_{x=-\infty}^{\infty} e^{-\frac{1}{2}(x^2+w^2+x^2-2wx)}\,dx$$

$$= \frac{1}{2\pi}\int_{x=-\infty}^{\infty} e^{-\frac{1}{2}\left(2x^2+\frac{w^2}{2}+\frac{w^2}{2}-2wx\right)}\,dx$$

$$= \frac{1}{2\pi}\int_{x=-\infty}^{\infty} e^{-\frac{w^2}{4}}\cdot e^{-\frac{1}{2}\left(2x^2+\frac{w^2}{2}-2wx\right)}\,dx$$

$$= \underbrace{\frac{1}{2\pi}e^{-\frac{w^2}{4}}\int_{x=-\infty}^{\infty} e^{-\frac{1}{2}\left(x\sqrt{2}-\frac{w}{\sqrt{2}}\right)^2}\,dx}_{①}$$

let $p = x\sqrt{2} - \frac{w}{\sqrt{2}} \Rightarrow dp = \sqrt{2}\,dx \Rightarrow dx = \frac{1}{\sqrt{2}}\,dp$

If $x = -\infty \Rightarrow p = \infty$ If $x = \infty \Rightarrow p = -\infty$

$$① \Rightarrow f_W(w) = \underbrace{\frac{e^{-\frac{w^2}{4}}}{2\pi}\,\frac{1}{\sqrt{2}}\int_{p=-\infty}^{\infty} e^{-\frac{p^2}{2}}}_{②}$$

let $\int_{p=-\infty}^{\infty} e^{-\frac{p^2}{2}}\,dp = 2\int_{0}^{\infty} e^{-\frac{p^2}{2}}\,dp$ ∵ Gaussian is even function

$$\frac{p^2}{2} = z \Rightarrow p = \sqrt{2z}$$

$$\frac{2p}{2}\,dp = dz \Rightarrow dz = \sqrt{2}\sqrt{z}\,dp \Rightarrow dz = \frac{1}{\sqrt{2}\sqrt{z}}\,dz$$

$$= 2\int_{z=-\infty}^{\infty} e^{-z}\frac{1}{\sqrt{2}\sqrt{z}}\,dz$$

$$= \frac{2}{\sqrt{2}}\int_{0}^{\infty} e^{-z}z^{-\frac{1}{2}}\,dz = \frac{2}{\sqrt{2}}\int_{0}^{\infty} e^{-z}z^{-\frac{1}{2}+1-1}\,dz$$

$$= \frac{2}{\sqrt{2}}\int_{0}^{\infty} e^{-z}z^{\frac{1}{2}-1}\,dz = 2\,\Gamma\left[\frac{1}{2}\right] = \frac{2}{\sqrt{2}}\sqrt{\pi}$$

$$\therefore \underbrace{\int_{p=-\infty}^{\infty} e^{-\frac{p^2}{2}}\,dp = \frac{2}{\sqrt{2}}\sqrt{\pi}}_{③}$$

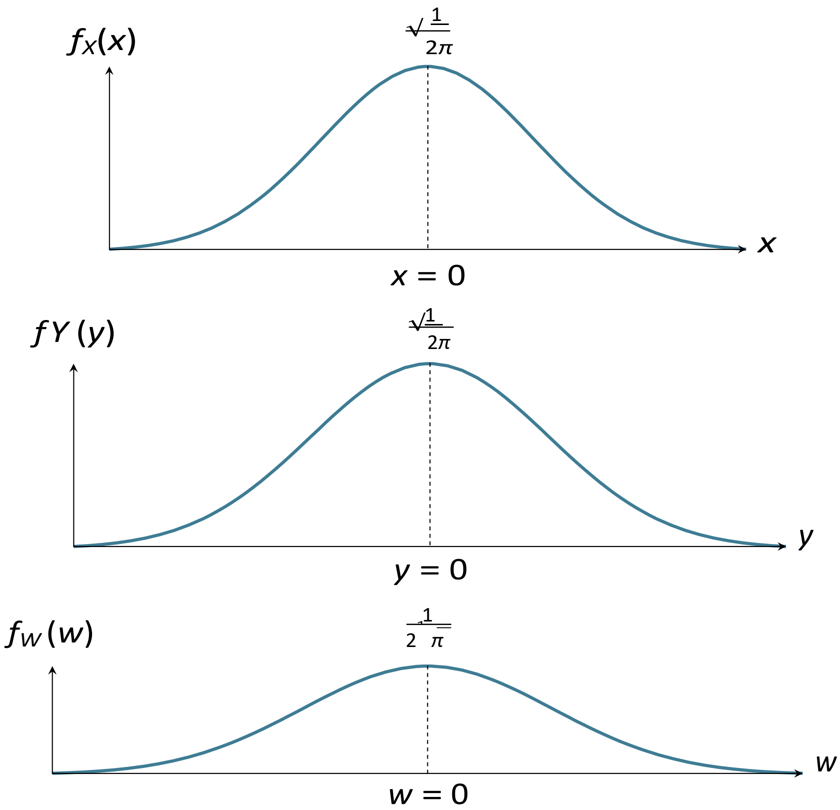
Substitute equation (3) in equation (2).

$$f_W(w) = \frac{e^{-\frac{w^2}{4}}}{2\pi} \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{4}}$$

$$\boxed{\therefore f_W(w) = \frac{1}{2\sqrt{\pi}} e^{-\frac{w^2}{4}}}$$

Conclusion: The addition, subtraction, multiplication and differentiation etc., of Gaussian random variable with different mean and variance.



5.8 Central limit theorem

The central limit theoram say that the probability density function of the sum of a large number of random variables approaches gaussian random variable.

Proof: (1) Equal distribution (2) Unequal distribution (not discussed)

(1) Equal distribution:

Let 'N' number of independent random variables $X_1, X_2, X_3 \ldots X_N$ with mean values $\overline{X_1}, \overline{X_2}, \overline{X_3} \ldots \overline{X_N}$; with variance $\sigma^2_{X_1}, \sigma^2_{X_3}, \sigma^2_{X_3} \ldots \sigma^2_{X_N}$.

Let all random variable are same or equal.

$$X_1 = X_2 = X_3 = \ldots = X_N = X \longrightarrow \text{same r.v}$$
$$\overline{X_1} = \overline{X_2} = \overline{X_3} = \ldots = \overline{X_N} = \overline{X} \longrightarrow \text{same mean}$$
$$\sigma^2_{X_1} = \sigma^2_{X_3} = \sigma^2_{X_3} = \ldots = \sigma^2_{X_N} = \sigma^2_X \longrightarrow \text{same variance}$$

Let $Z = \frac{Y - \overline{Y}}{\sigma_Y}$, This is taken to find PDF of sum of r.v.

Sum of random variables

$$Y = X_1 + X_2 + X_3 + \ldots + X_N = N X$$
$$\overline{Y} = \overline{X_1} + \overline{X_2} + \overline{X_3} + \ldots + \overline{X_N} = N \overline{X}$$
$$\sigma Y = \sigma^2_{X_1} + \sigma^2_{X_3} + \sigma^2_{X_3} + \ldots + \sigma^2_{X_N} = N \sigma^2_X$$
$$Z = \frac{NX - N\overline{X}}{\sigma_X \sqrt{N}}$$

We know that the characteristic function,

$$\Phi_Z(w) = E\left[e^{jw}\right] = \int_{x=-\infty}^{\infty} f_X(x)\, e^{jwx}\, dx$$

Find characteristic function of Z,

$$\Phi_Z(w) = E\left[e^{jwz}\right] = \int_{z=-\infty}^{\infty} f_Z(z)\, e^{jwz}\, dz$$
$$= E\left[e^{jw\frac{NX - N\overline{X}}{\sigma_X \sqrt{N}}}\right]$$
$$= E\left[e^{jwN\frac{X - \overline{X}}{\sqrt{N}\,\sigma_X}}\right]$$

Apply logarithm both sides and exponentiation of all r.v are equal

$$\ln \Phi_Z(w) = \ln \left(E\left[e^{jwN\frac{X-\bar{X}}{\sqrt{N}\sigma_X}} \right] \right) \longrightarrow ①$$

$$= \ln \left(E\left[e^{jw\frac{X-\bar{X}}{\sqrt{N}\sigma_X}} \right] \right)^N$$

$$= \frac{N}{n} \ln \left(E\left[e^{jw\frac{X-\bar{X}}{\sqrt{N}\sigma_X}} \right] \right) \longrightarrow ②$$

We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Consider,

$$E\left[e^{jw\frac{X-\bar{X}}{\sqrt{N}\sigma_X}} \right] = E\left[1 + \frac{jw\left(X-\bar{X}\right)}{\sqrt{N}\sigma_X} + j^2 \frac{w^2}{2N\sigma_X^2}\left(X-\bar{X}\right)^2 + \ldots\ldots \right]$$

$$= 1 + \frac{jw}{\sqrt{N}\sigma_X} E\left[X-\bar{X}\right] - \frac{w^2}{2N\sigma_X^2} E\left[X-\bar{X}\right]^2 + \ldots$$

$$= 1 + \frac{jw}{\sqrt{N}\sigma_X}(0) - \frac{w^2}{2N\sigma_X^2} \cdot \sigma_X^2 + \ldots\ldots$$

$$= 1 + \frac{w^2}{2N} + \underbrace{\ldots\ldots + \ldots}_{\text{Eliminate Higher order terms}}$$

$$\ln\left(E\left[e^{jw\frac{X-\bar{X}}{\sqrt{N}\sigma_X}} \right] \right) = \ln\left(1 - \frac{w^2}{2N}\right)$$

We know that $\ln(1-Z) = -\{Z + Z^2 + Z^3 + \ldots\}$

$$= -\left(\frac{w^2}{2N} + \frac{w^4}{4N^2} + \ldots\ldots \right)$$

From equation ② $\Rightarrow$

$$\ln \Phi_Z(w) = N\left(-\frac{w^2}{2N} - \frac{w^4}{4N^2} - \ldots\ldots \right)$$

$$= -\frac{w^2}{2} - \frac{w^4}{4N} - \ldots \quad \because \text{N is large or } \lim N \to \infty$$

$$\ln \Phi_Z(w) = -\frac{w^2}{2}$$

$$\boxed{\therefore \Phi_Z(w) = e^{-\frac{w^2}{2}}}$$

This is Gaussion PDF with unit variance and zero mean.

Application of central limit theorem: The bell shaped gaussian r.v help us in so many situations, the central limit theorem makes it possible to perform quick, accurate calculations, otherwise extremely complex and time consuming. In these calculations, the r.v of interest is a sum of other r.vs and we calculate the probabillities of event by referring to the Gaussian r.v.

Problem: Consider a communication system that transmits a data packets of 1024 bits.Each bit can be in error with probability of 10^{-2}. Find the (approximate) probability that more than 30 of the 1024 bits are in error.

Solution:

Let 'X' is a random vaiable, such that

$X_i = 1;$ if the i^{th} error

$X_i = 0;$ if not error

Given Data packet = 1024 bits.

$P(X_i = 1) = 10^{-2}$

$P(X_i = 0) = 1 - 10^{-2}$

The number of errors in the packet: $V = \sum_{i=1}^{1024} X_i$

Find $P(V > 30)$ =?, which means more than 30 errors.

$$P(V > 30) = \sum_{i=1}^{1024} 1024_{C_m}(10^{-2})^m (1 - 10^{-2})^{1024-m} \qquad \because \sum_{i=1}^{N} N_{P_k} p^k q^{N-k}$$

This calculation is time consuming. So, we apply central limit theorem, we can solve problem approximately

$$\overline{X_i} = 10^{-2} \times 1 + (1 - 10^{-2}) \times 0 = 10^{-2}$$

$$\overline{X_i^2} = 10^{-2} \times 1^2 + (1 - 10^{-2}) \times 0^2 = 10^{-2}$$

$$\sigma_i^2 = \overline{X_i^2} - (\overline{X_i})^2 = 0.0099$$

Based on central limmit theorem $V = \sum_{i=1}^{1024} X_i$ is approximately Gaussian with mean of $NP = \overline{V} = 1024 \times 10^{-2} = 10.24$

Variance $Npq = \sigma_V^2 = 1024 \times 0.0099 = 10.1376$

$$P(X > x) = Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$

$$\therefore P(V > 30) = Q\left(\frac{30 - 10.34}{\sqrt{10.1376}}\right)$$

$$= Q(6.20611)$$

$$= 1.925 \times 10^{-10}$$

Sum of several random variables:

Let 'N' number of random variables X_n; $n = 1, 2, 3, \ldots N$.

Whose PDF is $f_{X_N}(x_n)$; $n = 1, 2, 3, \ldots N$.

Sum of N random variable Y_N can be written as

$$Y_N = X_1 + X_2 + X_3 + \ldots + X_N$$

The probability density function of Y_N is convolution of individual probability density function. Thus,

$$f_{Y_N}(y_n) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * \ldots * f_{X_N}(x_N)$$

Problem: A random sample of size 100 is taken from a population whose mean is 60 and the variance is 400. Using central limit theorem , find the probability with which the mean of the sample will not differ from 60 by more than 4.

Problem: The life time of a certain band of an electrc bulb may be considered as a RV with mean 1200h and SD 250h.using central limit theorem,find the probability that the life time of 60 bulbs exceeds 1250h.

Problem: If V_i, $i = 1, 2, 3, 4, ..., 20$ are independent noise voltages received in an adder and V is the sum of the voltages received, find the probability that the total incoming voltage V exceeds 105, using the central limit theorem. Assume that each of the random variables Vi is uniformly distributed over (0, 10).

Problem: The life time of a particular variety of an electric bulb may be considered as a random variable with mean 1200h and SD 250h.Using central limit theorem, find the probability that the average life time of 60 bulbs exceeds 1250hours

Problem: If $X_1, X_2, ...X_n$ are Uniform variates with $mean = 2.5$ and $variance = 3/4$, use the central limit theorem to estimate $P(108 < Sn < 12.6)$,where $S_n = X_1 + X_2 + ...X_n$ and $n = 48$.

Problem: If $X_1, X_2, ...X_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit theorem to estimate $P(120 < Sn < 160)$, where $S_n = X_1 + X_2 + ...X_n$ and $n = 75$.

Problem: Describe Binomial $B(n, p)$ distribution and obtain the moment generating function. Hence compute (1). The first four moments and (2). The recursion relation for the central moments.

CHAPTER 6

Operations on the Multiple Random Variables

6.1 Joint Moment about the origin

Let 'X' and 'Y' are two random variables, the Joint moment is defined as

$$m_{nk} = E\ X^n\ Y^k\ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x^n y^k\, f_{XY}(x, y)\ dy\ dx$$

Here order of joint moment is "$n + k$".

1. If $k = 0$ then we will get only moment of r.v 'X'

$$m_{n0} = E\ X^n\ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x^n f_{XY}(x, y)\ dy\ dx = \int_{x=-\infty}^{\infty} x^n\ f_X(x)\ dx$$

2. If $n = 0$ then we will get only moment of r.v 'Y'

$$m_{0k} = E\ Y^k\ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} y^k f_{XY}(x, y)\ dy\ dx = \int_{y=-\infty}^{\infty} x^n\ f_Y(y)\ dx$$

3. If $n = 0$ and $k = 0$ then

$$m_{00} = E\ X^0 Y^0\ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y)\ dy\ dx = 1$$

   It is area of joint PDF i.e., equal to 1.

4. If $n = 1$ and $k = 0$ (or) $n = 0$ and $k = 1$, then we will get 1^{st} order moments.

   - The given m_{10} is an Expectation of r.v 'X'

$$m_{10} = E\ X^1\ Y^0\ = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x\, f_{XY}(x, y)\ dy\ dx = E[X]$$

- The given m_{01} is an Expectation of r.v 'Y'

$$m_{01} = E\left[X^0 Y^1\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{XY}(x, y)\ dy\ dx = E[Y]$$

5. If $n = 1$ and $k = 1$ (or) $n = 2$ and $k = 0$ (or) $n = 0$ and $k = 2$, then we will get second order moments.

(i). **Case 1:** $n = 1$ and $k = 1$

$$m_{11} = E[XY] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} xy\ f_{XY}(x, y)\ dy\ dx = R_{XY}$$

The second order moments $m_{11} = E[XY]$ is called the correlation of X and Y, denoted by R_{XY}

- If X and Y are independent then $f_{XY} = f_X(x) \cdot f_Y(y)$

$$m_{11} = R_{XY} = E[XY] = E[X] \cdot E[Y] = \int_{x=-\infty}^{\infty} xf_X(x)dx \cdot \int_{y=-\infty}^{\infty} yf_Y(y)dy$$

If $R_{XY} = 0$ then X and Y are orthogonal

If $R_{XY} = E[X]E[Y]$ then X and Y are Uncorrelated.

- If 'X' and 'Y' are independent then they are said to be uncorrelated. This is not true in general.
- If $R_{XY} = 0$ then X and Y are uncorrelated, they are called "orthogonal".

(ii). **Case 2:** $n = 2$ and $k = 0$, then we will get mean-square value of random variable 'X'.

$$m_{20} = E[X^2 Y^0] = E[X^2] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x^2\ f_{XY}(x, y)\ dy\ dx$$

(iii). **Case 3:** $n = 0$ and $k = 2$, then we will get mean-square value of random variable 'Y'.

$$m_{20} = E[X^0 Y^2] = E[Y^2] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} y^2\ f_{XY}(x, y)\ dy\ dx$$

NOTE:

- If X and Y are independent $f_{XY}(x, y) = f_X(x)f_Y(y)$
  Similarly, $R_{XY} = E[XY] = E[X]E[Y]$
- If X and Y are not independent then $E[XY] \neq E[X]E[Y]$
- If X and Y are mutually exclusive (or) orthogonal then $E[XY] = R_{XY} = 0$

6.2 Joint Central Moment (or) JointMoment about the Mean

The Joint cental moment of random variable X and Y is μ_{nk} can be written as

$$\mu_{nk} = E\left[(X - \overline{X})^n (Y - \overline{Y})^k\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})^n (y - \overline{Y})^k f_{XY}(x, y)\, dy\, dx$$

1. If $n = 0$ and $k = 0$ then

$$\mu_{00} = E\left[(X - \overline{X})^0 (Y - \overline{Y})^0\right] = E[1]$$
$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})^0 (y - \overline{Y})^0 f_{XY}(x, y)\, dy\, dx$$
$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx$$

$\therefore$ μ_{00} is area under the curve

2. i. If $n = 0$ and $k \neq 0$ then

$$\mu_{0k} = E\left[(Y - \overline{Y})^k\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})^0 (y - \overline{Y})^k f_{XY}(x, y)\, dy\, dx$$
$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (y - \overline{Y})^k f_{XY}(x, y)\, dy\, dx$$

ii. If $n \neq 0$ and $k = 0$ then

$$\mu_{n0} = E\left[(X - \overline{X})^n\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})^n (y - \overline{Y})^0 f_{XY}(x, y)\, dy\, dx$$
$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})^n f_{XY}(x, y)\, dy\, dx$$

3. i. If $n = 0$ and $k = 1$ then

$$\mu_{01} = E\left[(Y - \overline{Y})\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (y - \overline{Y}) f_{XY}(x, y)\, dy\, dx = 0$$

$$\therefore\ E[Y - \overline{Y}] = E[Y] - E[\overline{Y}] = \overline{Y} - \overline{Y} = 0$$

ii. If $n = 1$ and $k = 0$ then

$$\mu_{10} = E\left[(X - \overline{X})\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X}) f_{XY}(x, y)\, dy\, dx = 0$$

$$\therefore\ E[X - \overline{X}] = E[X] - E[\overline{X}] = \overline{X} - \overline{X} = 0$$

- If $n = 1$ and $k = 1$ then

$$\mu_{11} = E\left[(X - \overline{X})(Y - \overline{Y})\right] = C_{XY}$$

$$= \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X})(y - \overline{Y}) f_{XY}(x, y)\, dy\, dx$$

μ_{11} is second order central moment and it is called “Co-Variance” and is denoted by C_{XY}.

$$C_{XY} = \mu_{11} = E\left[(X - \overline{X})(Y - \overline{Y})\right]$$

$$= E\left[XY - X\overline{Y} - \overline{X}Y + \overline{X}\,\overline{Y}\right]$$

$$= E[XY] - E[X\overline{Y}] - E[\overline{X}Y] + E[\overline{X}\,\overline{Y}]$$

$$= R_{XY} - \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} + \overline{X}\,\overline{Y}$$

$$= R_{XY} - \overline{X}\,\overline{Y}$$

$$\therefore\ \boxed{C_{XY} = \mu_{11} = R_{XY} - \overline{X}\,\overline{Y} = R[XY] - E[X]E[Y]}$$

– If X and Y are independent and uncorrelated then $E[XY] = E[X]E[Y] = \overline{XY}$, and the $C_{XY} = 0$

– If X and Y are orthogonal r.vs then $C_{XY} = -E[X]E[Y]$

– If X and Y are orthogonal either X or Y has zero mean then $C_{XY} = 0$

4. i. If $n = 0$ and $k = 2$ then we will get variance ‘Y’

$$\mu_{02} = E\left[(Y - \overline{Y})^2\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (y - \overline{Y})^2 f_{XY}(x, y)\, dy\, dx$$

$$= \int_{y=-\infty}^{\infty} (y - \overline{Y})^2 f_{XY}(x, y)\, dy$$

ii. If $n = 2$ and $k = 0$ then we will get variance 'X'

$$\mu_{20} = E\left[(X - \overline{X})^2\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} (x - \overline{X}) f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=-\infty}^{\infty} (x - \overline{X})^2 f_{XY}(x, y)\, dx$$

$$Var(X) = \mu_{20} = m_{20} - m_{10}^2 = m_2 - m_1^2$$
$$Var(Y) = \mu_{02} = m_{02} - m_{00}^2 = m_2 - m_1^2$$

5. The normalized co-variance or normalized second order moment or correlation between X and Y is defined as

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\,\mu_{02}}} = \frac{C_{XY}}{\sigma_X\,\sigma_Y} = \frac{E\left[(X - \overline{X})(Y - \overline{Y})\right]}{\sigma_X\,\sigma_Y}$$

'ρ' is called corellation co-efficient of X and Y and it varies from -1 to $+1$.

$$\boxed{\therefore\ -1 \le \rho \le 1}$$

NOTE: The terminology, while widely used, is some what confusing, since orthogonal means zero correlation while uncorrelated means zero co-variance.

Problem: 1 Find all statistical parameters for given Joint PDF

$$f_{XY}(x, y) = \frac{xy}{9}; \quad 0 \le X \le 2; \quad 0 \le Y \le 3$$

Solution: Given Joint PDF

$$f_{XY}(x, y) = \begin{cases} \frac{xy}{9}; & 0 \le X \le 2; \quad 0 \le Y \le 3 \\ 0; & \text{otherwise} \end{cases}$$

The Joint moment $m_{nk} = E\left[X^n Y^k\right] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x^n y^k f_{XY}(x, y)\, dy\, dx$

1. If $n = 1$ and $k = 0$ then the mean value of r.v 'X' is

$$m_{10} = E[X^1 Y^0] = E[X] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} x \cdot \frac{xy}{9}\, dy\, dx$$

$$= \int_{x=0}^{2} \frac{x^2}{9}\left[\frac{y^2}{2}\right]_0^3 dx$$

$$= \int_{x=0}^{2} \frac{x^2}{9}\left[\frac{9}{2}\right] dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2} \times \frac{8}{3} = \frac{4}{3}$$

$$\boxed{\therefore\ E[X] = m_{10} = \frac{4}{3}}$$

2. If $n = 0$ and $k = 1$ then the mean value of r.v 'Y' is

$$m_{01} = E[X^0Y^1] = E[Y] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} yf_{XY}(x, y)\ dy\ dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} y \cdot \frac{xy}{9}\ dy\,dx$$

$$= \int_{x=0}^{2} \frac{x}{9}\left[\frac{y^3}{3}\right]_0^3 dx$$

$$= \int_{x=0}^{2} \frac{x}{9}\left[\frac{27}{3}\right] dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} = 2$$

$$\boxed{\therefore\ E[Y] = m_{01} = 2}$$

3. $\boxed{E[X]E[Y] = \frac{4}{3} \times 2 = \frac{8}{3}}$

4. If $n = 1$ and $k = 1$ then correlation

$$m_{11} = E[X^1Y^1] = R_{XY} = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} xyf_{XY}(x, y)\ dy\ dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} y \cdot \frac{xy}{9}\ dy\,dx$$

$$= \int_{x=0}^{2} \frac{x^2}{9}\left[\frac{y^3}{3}\right]_0^3 dx$$

$$= \int_{x=0}^{2} \frac{x^2}{9}\left[\frac{27}{3}\right] dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$\boxed{\therefore\ m_{11} = R_{XY} = \frac{8}{3}}$$

∵ If $m_{11} = R_{XY} = E[XY] = E[X]E[Y]$ then X and Y are independent. Here, $R_{XY} = E[X]E[Y]$ is satisfied. So, X and Y are independent.

5. If $n = 2$ and $k = 0$ then correlation

$$m_{20} = E[X^2Y^0] = E[X^2] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x^2 f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} x^2 \cdot \frac{xy}{9}\, dy\, dx$$

$$= \int_{x=0}^{2} \frac{x^3}{9}\left[\frac{y^2}{2}\right]_0^3 dx$$

$$= \int_{x=0}^{2} \frac{x^3}{9}\left[\frac{9}{2}\right] dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 = \frac{1}{2}\times\frac{16}{4} = 2$$

$$\boxed{\therefore\ m_{20} = E[X^2] = 2}$$

6. If $n = 0$ and $k = 2$ then correlation

$$m_{02} = E[X^0Y^2] = E[Y^2] = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} y^2 f_{XY}(x, y)\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} y^2 \cdot \frac{xy}{9}\, dy\, dx$$

$$= \int_{x=0}^{2} \frac{x}{9}\left[\frac{y^4}{4}\right]_0^3 dx$$

$$= \int_{x=0}^{2} \frac{x}{9}\left[\frac{81}{4}\right] dx$$

$$= \frac{9}{4}\left[\frac{x^2}{2}\right]_0^2 = \frac{9}{4} \times \frac{4}{2} = \frac{9}{2}$$

$$\boxed{\therefore m_{20} = E[Y^2] = \frac{9}{2}}$$

7. $\sigma_X^2 = m_2 - m_1^2 = E[X^2] - \left(E[X]\right)^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{18-16}{9} = \frac{2}{9}$

8. $\sigma_Y^2 = m_2 - m_1^2 = E[Y^2] - \left(E[Y]\right)^2 = \frac{9}{2} - (2)^2 = \frac{9-8}{2} = \frac{1}{2}$

9. Correlation:

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_{y=0}^{3} \frac{xy}{9}\, dy = \frac{x}{9}\left[\frac{y^2}{2}\right]_0^3 = \frac{x}{2}$$

$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0; & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_{y=0}^{3} \frac{xy}{9}\, dy = \frac{y}{9}\left[\frac{x^2}{2}\right]_0^2 = \frac{y}{9} \times \frac{4}{2} = \frac{2y}{9}$$

$$f_Y(y) = \begin{cases} \frac{2y}{9}; & 0 \le y \le 3 \\ 0; & \text{otherwise} \end{cases}$$

$$f_X(x) \cdot f_Y(y) = \frac{x}{2} \cdot \frac{2y}{9} = \frac{xy}{9} = f_{XY}(x, y)$$

Hence $f_{XY}(x, y) = f_X(x) f_Y(y)$

10. C_{XY} = co-variance = $\mu_{11} = R_{XY} - \overline{X}\,\overline{Y} = \frac{8}{3} - \frac{4}{3}(2) = 0$

$\therefore C_{XY} = 0$ then X and Y are independent.

11. Normalized co-variance: $\rho = \frac{\mu_{11}}{\sqrt{\mu_{02}\mu_{20}}} = \frac{C_{XY}}{\sigma_X \sigma_Y} = 0$

Problem 2: If mean and variance of random variable 'X' is 3 and 2 respectively. Find all statistical parameters of $Y = -6X + 22$

Solution: Given $Y = -6X + 22$

$$E[X] = \overline{X} = m_1 = m_{10} = 3$$

$$\sigma_X^2 = E[(X - \overline{X})^2] = \mu_{20} = 2$$

We know that

$$\sigma_X^2 = m_2 - m_1^2$$

$$\Rightarrow 2 = m_2 - (3)^2$$

$$\Rightarrow m_2 = 2 + 9 = 11$$

$$\boxed{\therefore\ m_2 = E[X^2] = m_{20} = 11}$$

1. Mean value of Y:

$$E[Y] = \overline{Y} = m_{01} = E[-6X + 22]$$
$$= -6E[X] + E[22]$$
$$= -6 \times 3 + 22 = 4$$

2. Mean square value of Y:

$$E[Y^2] = m_{02} = E[(-6X + 22)^2]$$
$$= E[36X^2 + 484 - 264X]$$
$$= 36E[X^2] + 484 - 264E[X]$$
$$= 36 \times 11 + 484 - 264 \times 3$$
$$= 396 + 484 - 792$$
$$= 88$$

3. Correlation:

$$R_{XY} = m_{11} = E[XY] = E[X(-6X + 22)]$$
$$= E[-6X^2 + 22X]$$
$$= -6E[X^2] + 22E[X]$$
$$= -6 \times 11 + 22 \times 3$$
$$= 0$$

$\therefore$ X and Y are orthogonal and not independent.

(or)

$R_{XY} = E[X]E[Y]$

$0 \quad 3 \times 4$

$0 \quad 12$

So, X and Y are uncorrelated

$E[XY] = E[X]E[Y]$

$0 \ /= 3 \times 4$

$0 \ /= 12$

So, X and Y are independent

4. Variance of Y

$$\mu_{02} = m_{02} - m_{01}^2$$
$$= 88 - (4)^2$$
$$= 88 - 16$$
$$= 72$$

Problem 3: Three statistical independent r.vs X_1, X_2, X_3 have mean values $\overline{X_1} = 3,\ \overline{X_2} = 6,\ \overline{X_1} = -2$. Find the mean values of the following functions.

1. $g(X_1, X_2, X_3) = X_1 + 3X_2 + 4X_3$
2. $g(X_1, X_2, X_3) = X_1X_2X_3$
3. $g(X_1, X_2, X_3) = -2X_1X_2 - 3X_1X_2 + 4X_2X_3$
4. $g(X_1, X_2, X_3) = X_1 + X_2 + X_3$

Solution:

1. $E\left[g(X_1, X_2, X_3)\right] = E[X_1] + 3E[X_2] + 4E[X_3]$

$$= 3 + 3 \times 6 + 4 \times (-2) = 13$$

2. $E\left[g(X_1, X_2, X_3)\right] = E[X_1]E[X_2]E[X_3]$

$$= 3 \times \times(-2) = -36$$

3. $E\left[g(X_1, X_2, X_3)\right] = -2E[X_1]E[X_2] - 3E[X_1]E[X_2] + 4E[X_2]E[X_3]$

$$= -2 \times 3 \times 6 + 3 \times 3 \times 2 + 4 \times 6 \times (-2)$$
$$= -36 + 18 - 48 = -66$$

3. $E\left[g(X_1, X_2, X_3)\right] = E[X_1] + E[X_2] + E[X_3]$

$$= 3 + 6 - 2 = 7$$

6.3 Properties of Co-Variance

1. Co-variance between X and Y is $C_{XY} = R_{XY} - \overline{X}\,\overline{X}$

Proof.

$$C_{XY} = E[(X - \overline{X})(Y - \overline{Y})]$$
$$= E[XY - X\overline{Y} - \overline{X}Y + \overline{X}\,\overline{Y}]$$
$$= E[XY] - \overline{Y}E[X] - \overline{X}E[Y] + \overline{X}\,\overline{Y}$$
$$= E[XY] - \overline{X}\,\overline{Y} - \overline{X}\,\overline{Y} + \overline{X}\,\overline{Y}$$

$$= E[XY] - \bar{X}\,\bar{Y}$$

□

2. If X and Y are independent then $C_{XY} = 0$

Proof.

$$C_{XY} = E[XY] - \bar{X}\,\bar{Y}$$
$$= E[XY] - E[X]E[Y]$$
$$= E[X]E[Y] - E[X]E[Y] \qquad \because X \text{ and } Y \text{ are independent}$$
$$= 0$$

□

3. Prove $Var(X + Y) = Var(X) + Var(Y) + C_{XY}$
and $Var(X - Y) = Var(X) + Var(Y) - 2C_{XY}$

$$(i)\ Var(X) = \sigma_X^2 = E[X^2] - \left(E[X]\right)^2$$

$$Var(X + Y) = E\left[(X + Y)^2\right] - \left(E[X + Y]\right)^2 \qquad \because \overline{X + Y} = E[X + Y]$$
$$= E[X^2 + Y^2 + 2XY] - \left(\overline{X + Y}\right)^2 \qquad \because E[X + Y] = E[X] + E[Y]$$
$$= E[X^2 + Y^2 + 2XY] - \left(\bar{X} + \bar{Y}\right)^2$$
$$= E[X^2 + Y^2 + 2XY] - \left(E[X] + E[Y]\right)^2$$
$$= E[X^2] + E[Y^2] + 2E[XY] - \left[\left(E[X]\right)^2 + \left(E[Y]\right)^2 + 2E[X]E[Y]\right]$$
$$= \left[E[X^2] - \left(E[X]\right)^2\right] + \left[E[Y^2] - \left(E[Y]\right)^2\right] + 2\left[E[XY] - E[X]E[Y]\right]$$
$$= Var(X) + Var(Y) + 2C_{XY}$$

$$\therefore\ Var(X + Y) = Var(X) + Var(Y) + 2C_{XY}$$

$\therefore\ Var(X + Y) = Var(X) + Var(Y);$ If X and Y are independent

$$(ii)\ Var(Y) = \sigma_Y^2 = E[Y^2] - \left(E[Y]\right)^2$$

$$Var(X - Y) = E\left[(X - Y)^2\right] - \left(E[X - Y]\right)^2 \qquad \because \overline{X - Y} = E[X - Y]$$
$$= E[X^2 + Y^2 - 2XY] - \left(\overline{X - Y}\right)^2 \qquad \because E[X - Y] = E[X] - E[Y]$$
$$= E[X^2 + Y^2 - 2XY] - \left(\bar{X} - \bar{Y}\right)^2$$
$$= E[X^2 + Y^2 - 2XY] - \left(E[X] - E[Y]\right)^2$$

$$= E[X^2] + E[Y^2] - 2E[XY] - \left[\left(E[X]\right)^2 + \left(E[Y]\right)^2 - 2E[X]E[Y] \right]$$

$$= E[X^2] + E[Y^2] - 2E[XY] - \left(E[X]\right)^2 - \left(E[Y]\right)^2 + 2E[X]E[Y]$$

$$= \left[E[X^2] - \left(E[X]\right)^2 \right] + \left[E[Y^2] - \left(E[Y]\right)^2 \right] - 2\left[E[XY] - E[X]E[Y] \right]$$

$$= Var(X) + Var(Y) - 2C_{XY}$$

$$\boxed{\therefore\ Var(X - Y) = Var(X) + Var(Y) - 2C_{XY}}$$

$$\boxed{\therefore\ Var(X - Y) = Var(X) + Var(Y);\quad \text{If } X \text{ and } Y \text{ are independent}}$$

4. $COV(aX, bY) = ab\ COV(X, Y)$ where a and b are constants.

$$COV(X, Y) = C_{XY} = \mu_{11} = E\left[(X - \overline{X})(Y - \overline{Y}) \right]$$

$$COV(aX, bY) = E\left[(aX - \overline{aX})(bY - \overline{bY}) \right]$$

$$= E\left[a(X - \overline{X})b(Y - \overline{Y}) \right]$$

$$= abE\left[(X - \overline{X})(Y - \overline{Y}) \right]$$

$$= ab\ COV(X, Y)$$

5. $COV(X + a, Y + b) = COV(X, Y)$ where a and b are constants.

$$COV(X, Y) = C_{XY} = \mu_{11} = E\left[(X - \overline{X})(Y - \overline{Y}) \right]$$

$$COV(X + a, Y + b) = E\left[(X + a - \overline{X + a})(Y + b - \overline{Y + b}) \right]$$

$$= E\left[(X + \not{a} - \overline{X} - \not{a})(Y + \not{b} - \overline{Y} - \not{b}) \right]$$

$$= E\left[(X - \overline{X})(Y - \overline{Y}) \right]$$

$$= COV(X, Y)$$

6. $COV(X + Y, Z) = COV(X, Z) + COV(Y, Z)$ where a and b are constants.

$$COV(X, Y) = C_{XY} = \mu_{11} = E\left[(X - \overline{X})(Y - \overline{Y}) \right]$$

$$COV(X + Y, Z) = E\left[(X + Y - \overline{X + Y})(Z - \overline{Z}) \right]$$

$$= E\left[(X + Y - \overline{X} - \overline{Y})(Z - \overline{Z}) \right]$$

$$= E\left[(X - \overline{X})(Z - \overline{Z}) \right] + E\left[(Y - \overline{Y})(Z - \overline{Z}) \right]$$

$$= COV(X, Z) + COV(Y, Z)$$

6.3.1 Theorems

Theorem 1: Expectation of sum of weighted random variables is equal to sum of weighted expectation or mean values.

Proof. Let a function with 'N' random variables $X_1, X_2, X_3, \ldots X_N$ their weights are $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_N$; where '$\alpha_i$' is constant.

Let Y be the sum of weighted random variables.

$$Y = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \ldots + \alpha_N X_N$$

$$= \sum_{i=1}^{N} \alpha_i X_i, \quad \text{where } \alpha_i \text{ is constant, Now}$$

$$E[Y] = E[\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \ldots + \alpha_N X_N]$$

$$= E[\alpha_1 X_1] + E[\alpha_2 X_2] + E[\alpha_3 X_3] + \ldots + E[\alpha_N X_N]$$

$$= \alpha_1 E[X_1] + \alpha_2 E[X_2] + \alpha_3 E[X_3] + \ldots + \alpha_N E[X_N] \quad \boxed{\because E[kX] = kE[X]}$$

$$= \sum_{i=1}^{N} \alpha_i E[X_i] = \sum_{i=1}^{N} \alpha_i \overline{X_i}$$

$$\boxed{\therefore E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \sum_{i=1}^{N} \alpha_i \overline{X_i}}$$

□

Theorem 2: Variance of sum of weighted random variables is equal to weighted sum of Variance of random variable (weights α_i^2).

Proof. Let a random variable $X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \ldots + \alpha_N X_N = \sum_{i=1}^{N} \alpha_i \overline{X_i}$

Expectation $E[X] = E[\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \ldots + \alpha_N X_N] = E\left[\sum_{i=1}^{N} \alpha_i X_i\right] = \overline{X}$

$$Var(X) = \sigma_X^2 = E[(X - \overline{X})^2]$$

$$Var\left[\sum_{i=1}^{N} \alpha_i X_i\right] = E\left[\left(\sum_{i=1}^{N} \alpha_i X_i - \sum_{i=1}^{N} \alpha_i \overline{X_i}\right)^2\right]$$

$$= E\left[\sum_{i=1}^{N} \left(\alpha_i X_i - \alpha_i \overline{X_i}\right)^2\right]$$

$$= E\left[\sum_{i=1}^{N} \alpha_i^2 (X_i - \overline{X_i})^2\right]$$

$$= \sum_{i=1}^{N} \alpha_i^2 E\left[(X_i - \overline{X_i})^2\right]$$

$$= \sum_{i=1}^{N} \alpha_i^2 \, Var(X_i)$$

$$\therefore Var\left(\sum_{i=1}^{N} \alpha_i X_i\right) = \sum_{i=1}^{N} \alpha_i^2 \, Var(X_i)$$

□

HW 1: Two r.v X and Y has the following Joint PDF

$$f_{XY}(x, y) = \begin{cases} 2 - x - y; & 0 \le X \le 1 \text{ and } 0 \le Y \le 1 \\ 0; & \text{otherwise} \end{cases}$$

Ans:

$f_X(x) = \frac{3}{2} - x$ $\quad m_{20} = \frac{1}{4}$ $\quad C_{XY} = \frac{-1}{144}$

$f_Y(y) = \frac{3}{2} - y$ $\quad m_{02} = \frac{1}{4}$ $\quad \mu_{20} = \frac{11}{72}$

$E[X] = \frac{5}{12}$ $\quad m_{11} = R_{XY} = \frac{1}{6}$ $\quad \mu_{02} = \frac{11}{72}$

$E[Y] = \frac{5}{12}$

HW 2: Two r.v X and Y has the following Joint PDF

$$f_{XY}(x, y) = \begin{cases} c(2x + y); & 0 \le X \le 2 \text{ and } 0 \le Y \le 3 \\ 0; & \text{otherwise} \end{cases}$$

Ans:

$C = \frac{1}{2}$; $\quad f_X(x) = \frac{2x}{7} - \frac{3}{14}$ $\quad E[X^2] = m_{20} = \frac{48}{21}$ $\quad C_{XY} = \frac{32}{29}$

$f_Y(y) = \frac{24+4}{14}$ $\quad E[Y^2] = m_{02} = \frac{6}{7}$ $\quad \mu_{20} = \frac{20}{7}$

$E[X] = \frac{8}{7}$ $\quad m_{11} = R_{XY} = 2$ $\quad \mu_{02} = \frac{51}{14}$

$E[Y] = \frac{4}{7}$

Problem 4: Find the Joint CDF and all statistical parameters of r.v X and Y. Whose Joint CDF is shown in the table.

(x_i, y_i)	(-1,0)	(0,0)	(0,2)	(1,-2)	(1,1)	(1,3)
$P(X_i, Y_j)$	0.1	0.2	0.1	0.3	0.2	0.1

Solution: Joint PDF

$$f_{XY}(x, y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} P(X = x_i, Y = y_j)\ \delta(x - x_i)\ \delta(y - y_j)$$

$$f_{XY}(x, y) = \sum_{i=-1}^{1} \sum_{j=-2}^{3} P(X = x_i, Y = y_j)\ \delta(x - x_i)\ \delta(y - y_j)$$

$$f_{XY}(x, y) = 0.1\delta(x + 1)\delta(y) + 0.2\delta(x)\delta(y) + 0.1\delta(x)\delta(y - 2)$$
$$+ 0.3\delta(x - 1)\delta(y + 2) + 0.2\delta(x - 1)\delta(y - 1) + 0.1\delta(x - 1)\delta(y - 3)$$

$$F_{XY}(x, y) = 0.1U(x + 1)U(y) + 0.2U(x)U(y) + 0.1U(x)U(y - 2)$$
$$+ 0.3U(x - 1)U(y + 2) + 0.2U(x - 1)U(y - 1) + 0.1U(x - 1)U(y - 3)$$

$$F_X(x) = F_{XY}(x, \infty)$$
$$= 0.1U(x + 1) + 0.2U(x) + 0.1U(x) + 0.3U(x - 1)$$
$$+ 0.2U(x - 1) + 0.1U(x - 1)$$

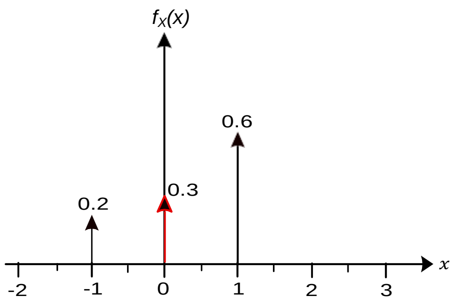
$$F_X(x) = 0.1U(x + 1) + 0.3U(x) + 0.6U(x - 1)$$

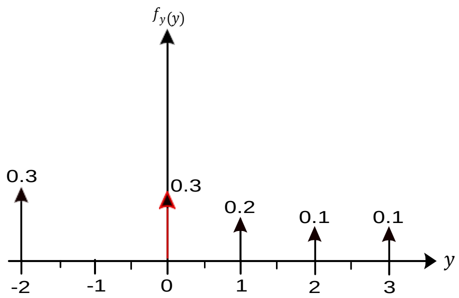
$$f_X(x) = \frac{d}{dx}F_X(x) = 0.1\delta(x + 1) + 0.3\delta(x) + 0.6\delta(x - 1)$$

$$F_Y(y) = F_{XY}(\infty, y)$$
$$= 0.1U(y) + 0.2U(y) + 0.1U(y - 2) + 0.3U(y + 2)$$
$$+ 0.2U(y - 1) + 0.1U(y - 3)$$

$$F_Y(y) = 0.3U(y + 2) + 0.3U(y) + 0.2U(y - 1) + 0.1U(y - 2) + 0.1U(y - 3)$$

$$f_Y(y) = 0.3\delta(y + 2) + 0.3\delta(y) + 0.2\delta(y - 1) + 0.1\delta(y - 2) + 0.1\delta(y - 3)$$





$$m_{10} = E[X] = \sum_{i=-\infty}^{\infty} x_i f_X(x_i)$$

$$= \sum_{i=-1}^{1} x_i f_X(x_i)$$

$$= (-1 \times 0.1) + (0 \times 0.3) + (1 \times 0.6)$$

$$= -0.4 + 0.6 = 0.5$$

$$m_{20} = E[X^2] = \sum_{i=-\infty}^{\infty} x_i^2 f_X(x_i)$$

$$= \sum_{i=-1}^{1} x^2_i f_X(x_i)$$

$$= (-1)^2 \times 0.1 + (0^2 \times 0.3) + (1^2 \times 0.6)$$

$$= 0.1 + 0.6 = 0.7$$

$$m_{01} = E[Y] = \sum_{j=-\infty}^{\infty} y_i f_Y(y_i)$$

$$= \sum_{j=-2}^{3} y_i f_Y(y_i)$$

$$= (-2 \times 0.3) + (0 \times 0.3) + (1 \times 0.2) + (2 \times 0.1) + (3 \times 0.1)$$

$$= -0.6 + 0.2 + 0.2 + 0.3 = 0.1$$

$$m_{02} = E[Y^2] = \sum_{j=-\infty}^{\infty} y_i^2 f_Y(y_i)$$

$$= \sum_{j=-2}^{3} y^2_i f_Y(y_i)$$

$$= (-2)^2 \times 0.3 + (0^2 \times 0.3) + (1^2 \times 0.2) + (2^2 \times 0.1) + (3^2 \times 0.1)$$

$$= 4 \times 0.3 + 0 + 0.2 + 4 \times 0.1 + 9 \times 0.1$$

$$= 1.2 + 0.2 + 0.4 + 0.9 = 2.7$$

$$\sigma_X^2 = m_2 - m^2{}_1$$

$$= 0.7 - 0.5^2$$

$$= 0.45 \Rightarrow \sigma_X = 0.6708$$

$$\sigma_Y^2 = m_2 - m^2{}_1$$

$$= 2.7 - 0.1^2$$

$$= 2.69 \Rightarrow \sigma_Y = 1.64$$

$$m_{11} = E[XY] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x_i y_j f_{XY}(x_i, y_j) = R_{XY}$$

$$R_{XY} = \sum_{i=-1}^{1} \sum_{j=-2}^{3} x_i y_j f_{XY}(x_i, y_j)$$

$$= 0 + 0 + 0 + (-2) \times 0.3 + 1 \times 0.2 + 3 \times 0.1$$

$$= -0.6 + 0.2 + 0.3 = -0.1$$

$$C_{XY} = R_{XY} - \overline{X}\,\overline{Y}$$

$$= -0.1 - (0.5 \times 0.1)$$

$$= -0.15$$

$$\rho = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{-0.15}{0.6708 \times 1.64} = -0.1365$$

6.4 Joint Characteristic function

Let X and Y are two random variables with Joint PDF $f_{XY}(x, y)$. The Joint characteristic function can ve written as

$$\Phi_{XY}(\omega_1, \omega_2) = E\left[e^{j\omega X + j\omega Y}\right] = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{XY}(x, y) e^{j\omega x + j\omega y} d\omega_2 \, d\omega_1$$

Take Fourier Transform both sides, then

$$f_{XY}(x, y) = \frac{1}{(2\pi)^2} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \Phi_{XY}(\omega_1, \omega_2) e^{-j\omega x - j\omega y} d\omega_2 \, d\omega_1$$

The Joint moment

$$m_{nk} = (-j)^{n+k} \left. \frac{d^{n+k}}{d\omega_1^n \, d\omega_1^k} \Phi_{XY}(\omega_1, \omega_2) \right|_{\omega_1=\omega_2=0}$$

6.4.1 Properties of Joint characteristic function

1. Marginal characteristic function

$$\Phi_X(\omega_1) = \Phi_{XY}(\omega_1, \omega_2)\big|_{\omega_2=0} = \Phi_{XY}(\omega_1)$$

$$\Phi_Y(\omega_2) = \Phi_{XY}(\omega_1, \omega_2)\big|_{\omega_1=0} = \Phi_{XY}(\omega_2)$$

2. Maximum value of $\Phi_{XY}(\omega_1, \omega_2) = 1$

3. $\Phi_{XY}(\omega_1, \omega_2) = \Phi_{\omega_1}\Phi_{\omega_2}$; if X and Y are independent.

4. $\Phi_{XY}(0, 0) = 1$; $\quad \Phi_{XY}(-\omega_1, -\omega_2) = \Phi^*_{XY}(\omega_1, \omega_2)$; $\quad |\Phi_{XY}(\omega_1, \omega_2)| \leq 1$

Problem 5: Find all statistical parameters of r.v X and Y, whose Joint characteristic function is given by $\Phi_{XY}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$

Solution: The Joint moment

$$m_{nk} = (-j)^{n+k} \frac{d^{n+k}}{d\omega_1^n \, d\omega_2^k} \Phi_{XY}(\omega_1, \omega_2)\Big|_{\omega_1=\omega_2=0}$$

1) when $n = 1, k = 0$ then

$$m_{10} = (-j)^{1+0} \frac{d^{1+0}}{d\omega_1^1 \, d\omega_2^0} \Phi_{XY}(\omega_1, \omega_2)\Big|_{\omega_1=\omega_2=0}$$

$$= (-j) \frac{d}{d\omega_1} e^{-2\omega_1^2 - 8\omega_2^2}\Big|_{\omega_1=\omega_2=0}$$

$$= (-j)\, e^{-8\omega_2^2} \frac{d}{d\omega_1} e^{-2\omega_1^2}\Big|_{\omega_1=\omega_2=0}$$

$$= (-j)\, e^{-8\omega_2^2} e^{-2\omega_1^2} (-4\omega_1)\Big|_{\omega_1=\omega_2=0}$$

$$\boxed{\therefore\; m_{01} = E[X] = \overline{X} = 0}$$

$$m_{01} = (-j)^{} \frac{}{d\omega_1^0 \, d\omega_2^1} \Phi_{XY}(\omega_1, \omega_2)\Big|$$

$.d\omega_2\, d\omega_1$ $\quad \omega_1=\omega_2=0$

$$= (-j)\, e^{-2\omega_1^2}\, d\, e^{-8\omega_2^2}.$$

2) when $n = 0, k = 1$ then

$$= (-j)\, e^{-2\omega_1^2}\, 0^{0+1}\, d^{0+1} \quad {}^{2}\ \omega_1=\omega_2=0$$

$$d \quad e^{-2\omega^2-8\omega^2} \quad .$$

$$= (-j)$$

2

$d\omega_2$ ____

$\omega = \omega = 0$

2

$\omega = \omega = 0$

$$\therefore\ m_{10} = E[Y] = \overline{Y} = 0$$

$$= (-\boldsymbol{j})\, e^{-2\omega^2}\, e^{-8\omega^2}\, (-16\omega\ ).$$

$$m_{02} = (-j)^{0+2} \frac{d^{0+2}}{d\omega_2^0\, d\omega_1^2} \Phi_{XY}(\omega_1, \omega_2) \Big|_{\omega_1=\omega_2=0}$$

$$= (-1) \frac{d^2}{d\omega_2^2} e^{-2\omega_1^2 - 8\omega_2^2} \Big|_{\omega_1=\omega_2=0}$$

$$= (-1)\, e^{-2\omega_1^2} \frac{d^2}{d\omega_2^2} e^{-8\omega_2^2} \Big|_{\omega_1=\omega_2=0}$$

$$= (-1)\, e^{-2\omega_1^2} \frac{d}{d\omega_2} e^{-8\omega_2^2}(-16\omega_2) \Big|_{\omega_1=\omega_2=0} \qquad \because d(uv) = u\, dv + v\, du$$

$$= (-1)(-16)\, e^{-2\omega_1^2} \frac{d}{d\omega_2} \left[\omega_2\, e^{-8\omega_2^2}\right] \Big|_{\omega_1=\omega_2=0}$$

$$= 16\, e^{-2\omega_1^2} \left[e^{-8\omega_2^2}(1) + \omega_2\, e^{-8\omega_2^2}(-16\omega_2)\right] \Big|_{\omega_1=\omega_2=0}$$

$$= 16 e^0 \left[e^0 + 0\right] = 16$$

4) when $n = 2, k = 0$ then

$$m_{20} = (-j)^{2+0} \frac{d^{2+0}}{d\omega_2^0\, d\omega_1^2} \Phi_{XY}(\omega_1, \omega_2) \Big|_{\omega_1=\omega_2=0}$$

$$= (-1) \frac{d^2}{d\omega_2^2} e^{-2\omega_1^2 - 8\omega_2^2} \Big|_{\omega_1=\omega_2=0}$$

$$= (-1)\, e^{-8\omega_2^2} \frac{d^2}{d\omega_1^2} e^{-2\omega_1^2} \Big|_{\omega_1=\omega_2=0}$$

$$= (-1)\, e^{-8\omega_2^2} \frac{d}{d\omega_1} e^{-2\omega_1^2}(-4\omega_1) \Big|_{\omega_1=\omega_2=0} \qquad \because d(uv) = u\, dv + v\, du$$

$$= (-1)(-4)\, e^{-8\omega_2^2} \frac{d}{d\omega_1} \left[\omega_1\, e^{-2\omega_1^2}\right] \Big|_{\omega_1=\omega_2=0}$$

$$= 4\, e^{-8\omega_2^2} \left[e^{-2\omega_1^2}(1) + \omega_1\, e^{-2\omega_1^2}(-4\omega_1)\right] \Big|_{\omega_1=\omega_2=0}$$

$$= 4 e^0 \left[e^0 + 0\right] = 4$$

5) when $n = 1, k = 1$ then

$$m_{11} = R_{XY} = (-j)^{1+1} \frac{d^{1+1}}{d\omega_1^1\, d\omega_2^1} \Phi_{XY}(\omega_1, \omega_2) \Big|_{\omega_1=\omega_2=0}$$

$$= (-1) \frac{d^2}{d\omega_1\, d\omega_2} e^{-2\omega_1^2 - 8\omega_2^2} \Big|_{\omega_1=\omega_2=0}$$

$$= (-1) \frac{d}{d\omega_1} e^{-2\omega_1^2} \frac{d}{d\omega_2} e^{-8\omega_2^2} \Big|_{\omega_1=\omega_2=0}$$

$$= -\left[e^{-2\omega_1^2}(-4\omega_1)\right]\left[e^{-8\omega_2^2}(-16\omega_2)\right] \Big|_{\omega_1=\omega_2=0}$$

$= (-1)(0)(0) = 0$

$\therefore m_{11} = R_{XY} = 0$; So, X and Y are orthogonal.

6) $Var(X) = \sigma_X^2 = E[X^2] - (\overline{X})^2 = 4 - 0 = 4$

7) $Var(Y) = \sigma_Y^2 = E[Y^2] - (\overline{Y})^2 = 16 - 0 = 16$

8) $C_{XY} = R_{XY} - \overline{XY} = 0 - 0 = 0$; So, X and Y are independent

9) $\rho = \dfrac{C_{XY}}{\sigma_X \sigma_Y} = \dfrac{0}{4 \times 16} = 0$

6.5 MGF of the sum of independent random variables

Moment generating functions (MGF) are particularly useful for analyzing sum of independent r.vs, because if X and Y are independent, the MGF of $W = X + Y$ is

$$\Phi_W(w) = E\left[e^{SX} e^{SY}\right] = E\left[e^{SX}\right] E\left[e^{SY}\right] = \Phi_X(S)\Phi_Y(S)$$

Problem: 6 If X and Y are independent r.v with PMF is

$$P_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.6, & x = 2 \\ 0.2, & x = 3 \\ 0, & x = \text{otherwise} \end{cases} \qquad P_X(x) = \begin{cases} 0.5, & y = -1 \\ 0.5, & y = 1 \\ 0, & y = \text{otherwise} \end{cases}$$

Find MGF of $W = X + Y$? What is $E\left[W^3\right]$ and $P_W(w)$ =?

Solution: If $W = X + Y$ then

$$\Phi_W(w) = E\left[e^{SX} e^{SY}\right] = E\left[e^{SX}\right] E\left[e^{SY}\right] = \Phi_X(S)\Phi_Y(S)$$

$\Phi_X(s) = 0.2e^s + 0.6e^{2s} + 0.3e^{3s}$

$\Phi_Y(s) = 0.5e^{-s} + 0.5e^s$

$\Phi_W(w) = \left[0.2e^s + 0.6e^{2s} + 0.3e^{3s}\right]\left[0.5e^{-s} + 0.5e^s\right]$

$= 0.1 + 0.3e^s + 0.2e^{2s} + 0.3e^{3s} + 0.1e^{4s}$

$$P_W(w) = \begin{cases} 0.1, & w=0 \\ 0.3, & w=1 \\ 0.2, & w=2 \\ 0.3, & w=3 \\ 0.1, & w=4 \end{cases} \qquad \text{(or)} P_W(w) = \begin{cases} 0.1, & w=0 \\ 0.3, & w=1,3 \\ 0.2, & w=2 \\ 0.1, & w=4 \end{cases}$$

$$E[W^3] = \frac{d^3}{dS^3}\Phi_W(w)\Big|_{s=0} = 0.3(1)^3e^s + 0.2(2^3)e^{2s} + 0.3(3^3)e^{3s} + 0.1(4^3)e^{4s} = 16.4$$

6.6 Characteristic function of sum of random variables

Let 'N' number of statistically independent r.vs $X_1, X_2, X_3, \ldots X_N$ with Joint PDF $f_{X_1,X_2,X_3,\ldots X_N}(x_1, x_2, x_3, \ldots x_N)$ then trhe characteristic function is

$$\Phi_{X^1+X^2+X^3+\ldots X^N}(\omega_1, \omega_2, \omega_3, \ldots \omega_N) = \prod_{i=1}^{N} \Phi_{Xi}(w_i)$$

Proof. For a r.v 'X' with PDF is $f_X(x)$ then characteristic function (CF) is

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{x=-\infty}^{\infty} f_X(x)e^{j\omega x}\,dx$$

For 'N' number of statistically independent r.vs PDF function is

$$f_{X_1,X_2,X_3,\ldots X_N}(x_1, x_2, x_3, \ldots x_N) = f_{X_1}(x_1)\cdot f_{X_2}(x_2)\cdot f_{X_3}(x_3)\ldots f_{X_N}(x_N)$$

The characteristic function of sum of r.vs

$$\Phi_{X_1+X_2+\ldots+X_N}(\omega_1,\omega_2,\ldots\omega_N) = \int_{x_1=-\infty}^{\infty}\int_{x_2=-\infty}^{\infty}\ldots\int_{x_N=-\infty}^{\infty} f_{X_1,X_2,\ldots X_N}(x_1,x_2,\ldots x_N)\,e^{j\omega_1x_1}e^{j\omega_2x_2}\ldots e^{j\omega_Nx_N}\,dx_1dx_2\ldots dx_N$$

$$= \int_{x_1=-\infty}^{\infty} f_{X^1}(x_1)e^{j\omega_1x_1}dx_1 \int_{x_2=-\infty}^{\infty} f_{X^2}(x_2)e^{j\omega_2x_2}dx_2 \ldots \int_{x_N=-\infty}^{\infty} f_{X_N}(x_3)e^{j\omega_N x_N}$$

$$= \Phi_{X_1}(\omega_1)\Phi_{X_2}(\omega_2)\ldots\Phi_{X_N}(\omega_N)$$

$$= \prod_{i=1}^{N}\Phi_{Xi}(\omega_i)$$

$$\therefore \Phi_{X_1+X_2+X_3+\ldots X_N}(\omega_1, \omega_2, \omega_3, \ldots \omega_N) = \prod_{i=1}^{N}\Phi_{Xi}(w_i)$$

□

6.7 Joint PDF of N-Gaussian random variables

Let 'N' number of Gaussian random variables with their PDF, mean, variance are

r.v	$\longrightarrow$	PDF	Mean	Variance
X_1	$\longrightarrow$	$f_{X_1}(x_1)$	$\overline{X_1}$	$\sigma^2_{X_1}$
X_2	$\longrightarrow$	$f_{X_2}(x_2)$	$\overline{X_2}$	$\sigma^2_{X_2}$
X_3	$\longrightarrow$	$f_{X_3}(x_3)$	$\overline{X_3}$	$\sigma^2_{X_3}$
.	.	.	.	.
X_N	$\longrightarrow$	$f_{X_N}(x_N)$	$\overline{X_N}$	$\sigma^2_{X_N}$

The Joint PDF of 'N' Gaussian r.v can be written as

Joint PDF:

$$f_{X_1X_2\ldots X_N}(x_1, x_2, \ldots x_N) = \frac{\left|[C_X]^{-1}\right|^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} Exp\left\{-\frac{1}{2}[x - \overline{X}]^T [C_X]^{-1} [x - \overline{X}]\right\} \qquad (6.1)$$

where we define matrices is Co-variance matrix: $[C_X]_{N\times N}$ and $[x - \overline{X}]_{N\times 1}$

The elements of co-variance matrix of 'N' r.v are given by

$$C_{X_iX_j} = E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right] = \sigma^2_{X_i} = \sigma^2_{X_j}; \quad \text{if } i = j$$

$$= \rho\sigma_{X_i}\sigma_{X_j}; \quad \text{if } i \neq j$$

$\because C_{XY} = E[(X - \overline{X})(Y - \overline{Y})]$	$\because \rho = \dfrac{C_{XY}}{\sigma_X\sigma_Y}$

$$\left[C_X\right]_{N\times N} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} & C_{X_1X_3} & \ldots & C_{X_1X_N} \\ C_{X_2X_1} & C_{X_2X_2} & C_{X_2X_3} & \ldots & C_{X_2X_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{X_NX_1} & C_{X_NX_2} & C_{X_NX_3} & \ldots & C_{X_NX_N} \end{bmatrix}_{N\times N} = C_{X_iX_j}$$

$$\left[x - \overline{X}\right]_{N\times 1} = \begin{bmatrix} x_1 - \overline{X_1} \\ x_2 - \overline{X_2} \\ \vdots \\ x_N - \overline{X_N} \end{bmatrix}_{N\times 1}$$

Note: $[\cdot]^T \rightarrow$ Matrix transpose; $[\cdot]^{-1} \rightarrow$ Matrix inverse; $|[\cdot]| \rightarrow$ Matrix determinent

Fir $N = 2$, from equation (6.1)

$$f_{X_1X_2}(x_1, x_2) = \frac{\left|[C_X]^{-1}\right|^{\frac{1}{2}}}{2\pi} Exp\left\{-\frac{1}{2}[x-\bar{X}]^T\,[C_X]^{-1}\,[x-\bar{X}]\right\} \qquad (6.2)$$

where

$$C_X = \left[C_X\right]_{2\times 2} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} \\ C_{X_2X_1} & C_{X_2X_2} \end{bmatrix}_{2\times 2} = \begin{bmatrix} \sigma^2_{X_1} & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma^2_{X_2} \end{bmatrix}_{2\times 2}$$

We know that matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\left[C_X\right]^{-1} = \frac{1}{\sigma^2_{X_1}\sigma^2_{X_2} - \rho^2\sigma^2_{X_1}\sigma^2_{X_2}}\begin{bmatrix} \sigma^2_{X_2} & -\rho\sigma_{X_1}\sigma_{X_2} \\ -\rho\sigma_{X_1}\sigma_{X_2} & \sigma^2_{X_1} \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)(\sigma^2_{X_1}\sigma^2_{X_2})}\begin{bmatrix} \sigma^2_{X_2} & -\rho\sigma_{X_1}\sigma_{X_2} \\ -\rho\sigma_{X_1}\sigma_{X_2} & \sigma^2_{X_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(1-\rho^2)\sigma^2_{X_1}} & -\frac{\rho}{(1-\rho^2)\sigma_{X_1}\sigma_{X_2}} \\ -\frac{\rho}{(1-\rho^2)\sigma_{X_1}\sigma_{X_2}} & \frac{1}{(1-\rho^2)\sigma^2_{X_2}} \end{bmatrix}$$

$$\left|\left[C_X\right]^{-1}\right| = \frac{1}{(1-\rho^2)^2\sigma^2_{X_1}\sigma^2_{X_2}} - \frac{(-\rho)^2}{(1-\rho^2)^2\sigma^2_{X_1}\sigma^2_{X_2}}$$

$$= \frac{(1-\rho^2)}{(1-\rho^2)^2\sigma^2_{X_1}\sigma^2_{X_2}}$$

$$= \frac{1}{(1-\rho^2)\sigma^2_{X_1}\sigma^2_{X_2}}$$

$$\left|\left[C_X\right]^{-1}\right|^{\frac{1}{2}} = \left(\frac{1}{(1-\rho^2)\sigma^2_{X_1}\sigma^2_{X_2}}\right)^{\frac{1}{2}} = \frac{1}{\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} \qquad (6.3)$$

$$\left[x-\bar{X}\right]_{2\times 1} = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \end{bmatrix}_{2\times 1}; \qquad \left[x-\bar{X}\right]^T = \left[(x_1-\bar{X}_1) \quad (x_2-\bar{X}_2)\right]_{1\times 2} \qquad (6.4)$$

Substitute equation (6.3) and (6.4) in equation (6.8), then

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} Exp\left\{-\frac{1}{2}\left[(x_1-\bar{X}_1) \quad (x_2-\bar{X}_2)\right]_{1\times 2} \times \begin{bmatrix} \frac{1}{(1-\rho^2)\sigma^2_{X_1}} & -\frac{\rho}{(1-\rho^2)\sigma_{X_1}\sigma_{X_2}} \\ -\frac{\rho}{(1-\rho^2)\sigma_{X_1}\sigma_{X_2}} & \frac{1}{(1-\rho^2)\sigma^2_{X_2}} \end{bmatrix}\right.$$

$$-\frac{\rho(x_2-X_2)}{} + \frac{x_2-X_2}{}$$

2×1

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}$$

$$\times\, Exp\left(-\frac{1}{2}\left[\frac{(x_1-\overline{X_1})^2}{(1-\rho^2)\sigma_{X_1}^2} - \frac{2\rho(x_1-\overline{X_1})(x_2-\overline{X_2})}{(1-\rho^2)\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\overline{X_2})^2}{(1-\rho^2)\sigma_{X_2}^2}\right]\right)$$

6.7.1 Properties

1. Maximum value occurs at $x_1 = \overline{X_1}$ and $x_2 = \overline{X_2}$ i.e.,

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}$$

2. If $\rho = 0$ then independent.

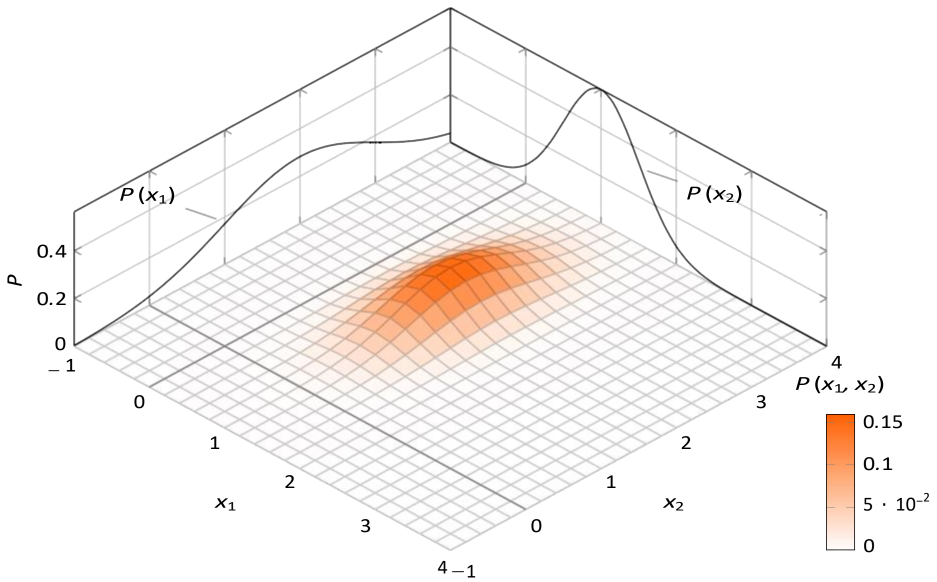
$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}}\, e^{-\frac{1}{2}\left[\frac{(x_1-\overline{X_1})^2}{\sigma_{X_1}^2} + \frac{(x_2-\overline{X_2})^2}{\sigma_{X_2}^2}\right]}$$

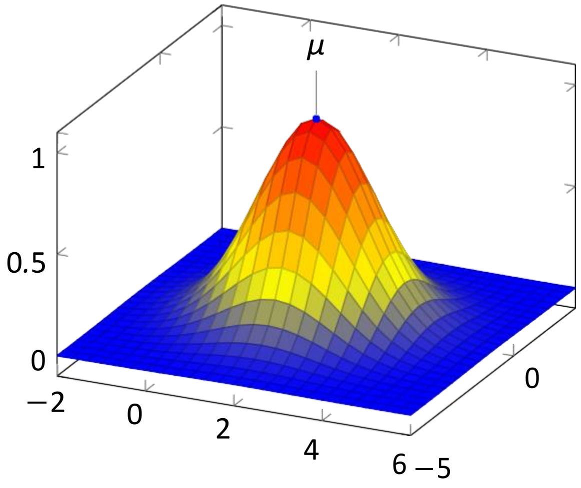
$$= \frac{1}{\sqrt{2\pi}\,\sigma_{X_1}}\, e^{-\frac{1}{2}\left[\frac{(x_1-\overline{X_1})^2}{\sigma_{X_1}^2}\right]} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_{X_2}}\, e^{-\frac{1}{2}\left[\frac{(x_2-\overline{X_2})^2}{\sigma_{X_2}^2}\right]}$$

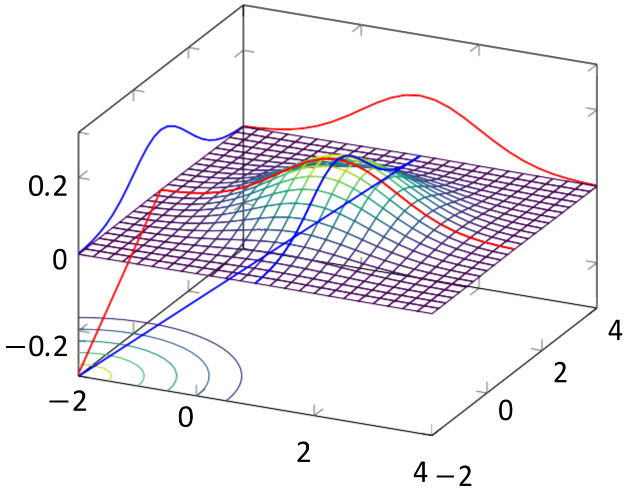
$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

$$\boxed{\therefore f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)}$$

The graph for the above expression is





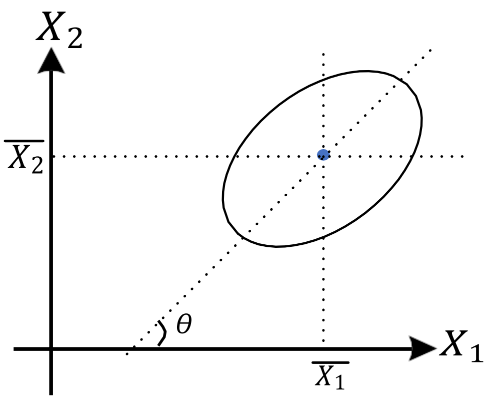


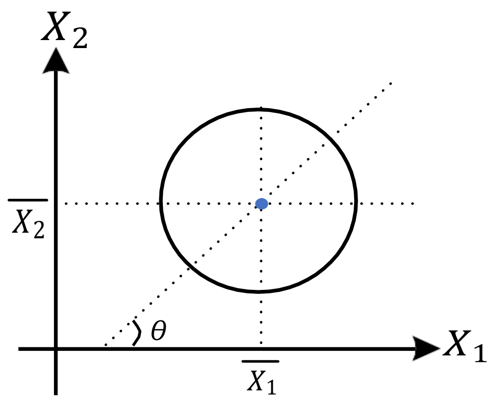
3. The locus point between X_1 and X_2 is shown in Fig.(a)
   The locus point is a elipse between X_1 and X_2. If $\rho = 0$,and $\sigma_{X_1} = \sigma_{X_2}$ then PDF is

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\,\sigma_X^2}\, e^{-\frac{1}{2}\left[\frac{(x_1-\overline{X}_1)^2+(x_2-\overline{X}_2)^2}{\sigma_X^2}\right]}$$

   and the locus between X_1 and X_2 is a circle as shown in Fig. (b)

   If $\rho = \pm 1$ and $\sigma_{X_1} = \sigma_{X_2}$ then the circle will be rotated of $\vartheta = \pm\frac{\pi}{4}$ to get independent random variables.





4. The expression for ϑ can be written as

$$\vartheta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_{X_1}\sigma_{X_2}}{\sigma_{X_1}^2 - \sigma_{X_2}^2}\right)$$

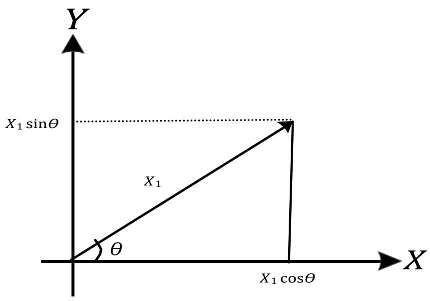
Note:

- $\rho = 0$ then independent
- $\rho = \pm 1$ is also independent but depends on ϑ values they are are independent.

Problem: Two random variables given by $Y_1 = X_1 \cos\vartheta + X_2 \sin\vartheta$ and $Y_2 = -X_1 \sin\vartheta + X_2 \cos\vartheta$

1. Find co-variance Y_1 and Y_2.
2. For what value of ϑ, the random variables Y_1 and Y_2 are uncorrelated.

Solution:



Let $\overline{Y_1}$ and $\overline{Y_2}$ are the means of r.vs Y_1 and Y_2 respectively and ϑ is called angle of rotations and X_1 and X_2 are Gaussian random variables.

Mean value of Y_1 is $\overline{Y_1} = \overline{X_1}\cos\vartheta + \overline{X_2}\sin\vartheta$

Mean value of Y_2 is $\overline{Y_2} = -\overline{X_1}\sin\vartheta + \overline{X_2}\cos\vartheta$

(1.)The Co-variance between Y_1 and Y_2 can be written as

$$C_{Y_1Y_2} = E\Big[(Y_1 - \overline{Y_1})(Y_2 - \overline{Y_2})\Big]$$

$$= E\Big[(X_1\cos\vartheta + X_2\sin\vartheta) - (\overline{X_1}\cos\vartheta + \overline{X_2}\sin\vartheta)\times (-X_1\sin\vartheta + X_2\cos\vartheta) - (-\overline{X_1}\sin\vartheta + \overline{X_2}\cos\vartheta)\Big]$$

$$= E\Big[(X_1\cos\vartheta + X_2\sin\vartheta - \overline{X_1}\cos\vartheta - \overline{X_2}\sin\vartheta)\times (-X_1\sin\vartheta + X_2\cos\vartheta + \overline{X_1}\sin\vartheta - \overline{X_2}\cos\vartheta)\Big]$$

$$= E\Big[\cos\vartheta[X_1 - \overline{X_1}] + \sin\vartheta[X_2 - \overline{X_2}] \times \sin\vartheta[-X_1 + \overline{X_1}] + \cos\vartheta[X_2 - \overline{X_2}]\Big]$$

$$= E\Big[\cos^2\vartheta(X_1 - \overline{X_1})(X_2 - \overline{X_2}) + \cos\vartheta\sin\vartheta(\overline{X_1} - X_1)(X_1 - \overline{X_1}) + \sin^2\vartheta(-X_2 + \overline{X_2})(\overline{X_1} - X_1) + \cos\vartheta\sin\vartheta(X_2 - \overline{X_2})(X_2 - \overline{X_2})\Big]$$

$$= E\Big[\cos^2\vartheta(X_1 - \overline{X_1})(X_2 - \overline{X_2}) - \cos\vartheta\sin\vartheta(X_1 - \overline{X_1})^2 - \sin^2\vartheta(X_1 - \overline{X_1})(X_2 - \overline{X_2}) + \cos\vartheta\sin\vartheta(X_2 - \overline{X_2})^2\Big]$$

$$= E\Big[(X_1 - \overline{X_1})(X_2 - \overline{X_2})\Big]\cos^2\vartheta - E\Big[(X_1 - \overline{X_1})^2\Big]\cos\vartheta\sin\vartheta$$

$$-E\left[(X_1-\overline{X_1})(X_2-\overline{X_2})\right]\sin^2\vartheta + E\left[(X_2-\overline{X_2})^2\right]\cos\vartheta\sin\vartheta$$

$$= C_{X_1X_2}\cos^2\vartheta - \sigma^2_{X_1}\cos\vartheta\sin\vartheta - C_{X_1X_2}\sin^2\vartheta + \sigma^2_{X_2}\cos\vartheta\sin\vartheta$$

$$= C_{X_1X_2}\left(\cos^2\vartheta - \sin^2\vartheta\right) + \left(\sigma^2_{X_2} - \sigma^2_{X_2}\right)\frac{2\sin\vartheta\cos\vartheta}{2}$$

$$= \rho\,\sigma_{X_1}\sigma_{X_2}\cos 2\vartheta + \left(\sigma^2_{X_2} - \sigma^2_{X_1}\right)\frac{\sin 2\vartheta}{2}$$

$$\boxed{\therefore C_{Y_1Y_2} = \rho\,\sigma_{X_1}\sigma_{X_2}\cos 2\vartheta + \left(\sigma^2_{X_2} - \sigma^2_{X_1}\right)\frac{\sin 2\vartheta}{2}}$$

(2.) Y_1 and Y_2 are uncorrelated if $C_{X_1X_2} = 0$, independent

$$\left(\sigma^2_{X_1} - \sigma^2_{X_2}\right)\frac{\sin 2\vartheta}{2} = \rho\,\sigma_{X_1}\sigma_{X_2}\cos 2\vartheta$$

$$\frac{\sin 2\vartheta}{\cos 2\vartheta} = \frac{2\rho\,\sigma_{X_1}\sigma_{X_2}}{\sigma^2_{X_1} - \sigma^2_{X_2}}$$

$$\tan 2\vartheta = \frac{2\rho\,\sigma_{X_1}\sigma_{X_2}}{\sigma^2_{X_1} - \sigma^2_{X_2}}$$

$$2\vartheta = \tan^{-1}\left(\frac{2\rho\,\sigma_{X_1}\sigma_{X_2}}{\sigma^2_{X_1} - \sigma^2_{X_2}}\right)$$

$$\boxed{\vartheta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\,\sigma_{X_1}\sigma_{X_2}}{\sigma^2_{X_1} - \sigma^2_{X_2}}\right)}$$

$\rho = \pm 1$ and independent then at ϑ =? is uncorrelated

Problem: Two Gaussian r.vs X_1 and X_2 have variance $\sigma^2_{X_1} = 9$; $\sigma^2_{X_2} = 4$ respectively. It is known that a coordinate rotation by an angle $\vartheta = \frac{\pi}{8}$ results new r.vs Y_1 and Y_2 such that they are independent. What is the ρ value?

Solution:

$$\vartheta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\,\sigma_{X_1}\sigma_{X_2}}{\sigma^2_{X_1} - \sigma^2_{X_2}}\right)$$

$$\frac{\pi}{8} = \frac{1}{2}\tan^{-1}\left(\frac{2\rho(3)(2)}{9-4}\right)$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{12\rho}{5}\right)$$

$$\frac{12}{5}\rho = \tan\frac{\pi}{4} = 1$$

$$\rho = \frac{5}{12} \qquad \boxed{\therefore \rho = 0.416}$$

6.8 Linear Transformation of Gaussian random variable

Let $X_1, X_2, X_3, \ldots X_N$ are Gaussian random variables, then their Joint PDF is

$$f_{X_1X_2\ldots X_N}(x_1, x_2, \ldots x_N) = \frac{\left|[C_X]^{-1}\right|^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} Exp\left\{-\frac{1}{2}[x-\overline{X}]^T [C_X]^{-1}[x-\overline{X}]\right\}$$

where
$$C_X = C_{X_iX_j} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} & C_{X_1X_3} & \ldots & C_{X_1X_N} \\ C_{X_2X_1} & C_{X_2X_2} & C_{X_2X_3} & \ldots & C_{X_2X_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{X_NX_1} & C_{X_NX_2} & C_{X_NX_3} & \ldots & C_{X_NX_N} \end{bmatrix}_{N\times N}$$

$$\left[x-\overline{X}\right] = \begin{bmatrix} x_1 - \overline{X_1} \\ x_2 - \overline{X_2} \\ \vdots \\ x_N - \overline{X_N} \end{bmatrix}_{N\times 1}$$

The linear transformation of $X_1, X_2, \ldots X_N$ are the new r.vs $Y_1, Y_2, \ldots Y_N$ are

$$Y_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1N}X_N$$

$$Y_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2N}X_N$$

$$\vdots$$

$$Y_N = a_{N1}X_1 + a_{N2}X_2 + \ldots + a_{NN}X_N$$

$$f_{Y_1Y_2\ldots Y_N}(y_1, y_2, \ldots y_N) = \frac{\left|[C_Y]^{-1}\right|^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} Exp\left\{-\frac{1}{2}[y-\overline{Y}]^T [C_Y]^{-1}[y-\overline{Y}]\right\}$$

where $C_Y = T\ C_X\ T^T$

C_X = Co-variance Matrix of X and T = Transformation matrix

Matrix representation

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{bmatrix}_{N\times N}}_{T=\text{Transformation matrix}} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}_{N\times 1}$$

Problem : A Gaussian random variable X_1 and X_2 for which $\overline{X_1} = 2$, $\sigma^2_{X_1} = 9$; $\overline{X_2} = -1$, $\sigma^2_{X_2} = 4$; and $C_{X_1X_2} = -3$ are transformed to new r.v Y_1 and Y_2 according to $Y_1 = -X_1 + X_2$, $Y_2 = -2X_1 - 3X_2$

Find (a) $\overline{X_1^2}$, $\overline{X_2^2}$, $\rho_{X_1X_2}$ (b) $\sigma^2_{Y_1}$, $\sigma^2_{Y_2}$, $\rho_{Y_1Y_2}$, $E[Y_1]$, $E[Y_2]$, $\overline{Y_1^2}$, $\overline{Y_2^2}$

(c) $f_{X_1X_2}(x_1, x_2)$, $f_{Y_1Y_2}(y_1, y_2)$

Solution: Given data

$$\overline{X_1} = E[X_1] = 2 \qquad \overline{X_2} = E[X_2] = -1$$

$$\sigma^2_{X_1} = E[(x_1 - \overline{X_1})^2] = 9 \qquad \sigma^2_{X_2} = E[(x_2 - \overline{X_2})^2] = 4$$

$$\sigma_{X_1} = 3 \qquad \sigma_{X_2} = 2$$

$$Y_1 = -X_1 + X_2 \qquad Y_2 = -2X_1 - 3X_2$$

$$C_{X_1X_2} = C_{X_2X_1} = -3 \qquad \rho_{X_1X_2} = \frac{C_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}} = \frac{-3}{3\times 2} = -0.5$$

$$\boxed{\rho_{X_1X_2} = -\frac{1}{2} = -0.5}$$

(a) we know that $\sigma^2_X = m_2 - m_1^2 = \overline{X^2} - (\overline{X})^2$

$$\overline{X_1^2} = \sigma^2_{X_1} + (\overline{X_1})^2 = 3^2 + (2)^2 = 13$$

$$\overline{X_2^2} = \sigma^2_{X_2} + (\overline{X_2})^2 = 2^2 + (-1)^2 = 5 \qquad \boxed{\therefore \overline{X_1^2} = 13 \quad \overline{X_2^2} = 5}$$

(b) $\overline{Y_1^2} = \sigma^2_{Y_1} + (\overline{Y_1})^2$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

we know that

where $C_X = C_{X_iX_j} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} \\ C_{X_2X_1} & C_{X_2X_2} \end{bmatrix} = \begin{bmatrix} \sigma^2_{X_1} & C_{X_1X_2} \\ C_{X_2X_1} & \sigma^2_{X_2} \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}$

If $i = j$; $C_{X_iX_j} = \sigma^2_{X_i} = \sigma^2_{X_j}$ and If $i \neq j$; $C_{X_iX_j} = \rho\, \sigma_{X_2}\sigma_{X_2}$

we know that linear transformation of Gaussian r.v co-variance matrix $C_Y = [T][C_X][T]^T$

$$C_Y = C_{Y_iY_j} = \begin{bmatrix} \sigma^2_{Y_1} & C_{Y_1Y_2} \\ C_{Y_2Y_1} & \sigma^2_{Y_2} \end{bmatrix} = [T][C_X][T]^T \tag{6.5}$$

$$C_Y = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}$$

$$
= \begin{bmatrix} (-1)(9)+1(-3) & (-1)(-3)+1(4) \\ (-2)(9)+(-3)(-3) & (-2)(-3)+(-3)(4) \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}
$$

$$
= \begin{bmatrix} -12 & 7 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}
$$

$$
= \begin{bmatrix} -12(-1)+7(1) & -12(-2)+7(-3) \\ -9(-1)+(-6)(1) & -9(-2)+(-6)(-3) \end{bmatrix}
$$

$$
= \begin{bmatrix} 12+7 & 24-21 \\ 9-6 & 18+18 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix}
$$

By comparing this matrix to equation (6.5)

$\sigma^2_{Y_1} = 19; \quad \sigma^2_{Y_2} = 19; \quad C_{Y_1Y_2} = C_{Y_2Y_1} = 3$

$$
\rho_{Y_1Y_2} = \frac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{3}{\sqrt{19}\times\sqrt{36}} = 0.1147
$$

$\therefore \rho_{Y_1Y_2} = 0.1147$

$$
Y_1 = -X_1 + X_2 \qquad\qquad Y_2 = -2X_1 - 3X_2
$$

$$
E[Y_1] = -E[X_1] + E[X_2] \qquad\qquad E[Y_2] = -2E[X_1] - 3E[X_2]
$$

$$
= -2 + (-1) \qquad\qquad = -2(2) - 3(-1)
$$

$$
= -3 \qquad\qquad = -1
$$

$\therefore E[Y_1] = -3; \quad E[Y_2] = -1$

$$
\overline{Y_1^2} = \sigma^2_{Y_1} + (\overline{Y_1})^2 = 19 + (-3)^2 = 28
$$

$$
\overline{Y_2^2} = \sigma^2_{Y_2} + (\overline{Y_2})^2 = 36 + (-1)^2 = 37
$$

$\therefore \overline{Y_1^2} = 28; \quad \overline{Y_2^2} = 37$

(f)

$$
f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\,[C_X]^{\frac{1}{2}}}\, Exp\left[-\frac{1}{2}[x-\bar{X}]^{T}[C_X]^{-1}[x-\bar{X}]\right]
$$

$$
C_X = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma^2_{X_1} & C_{12} \\ C_{21} & \sigma^2_{X_2} \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}
$$

$$
[C_X]^{-1} = \frac{1}{9(4) - (-3)(-3)} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} \frac{4}{27} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix}
$$

$$\frac{4}{27} \quad \frac{1}{9} \qquad \frac{4}{27} \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{9}$$

$$[C_X]^{-1} = \begin{vmatrix} \frac{4}{27} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{3} \end{vmatrix} = \frac{4}{27} \times \frac{1}{3} - \frac{1}{9} \times \frac{1}{9} = 0.03703$$

$$\frac{1}{2}$$

$$x$$

$$[C_X]^{-1 \; \frac{1}{2}} = 0.19245$$

$$(x_1 - 2)\underline{4} + (x_2 + 1)\underline{1} \quad x_1 - 2$$

$$(x_1 - 2)\frac{1}{9} + (x_2 + 1)\underline{1}$$

$$\left[x - X\right]^T \left[C_x\right]^{-1} \left[x - X\right]$$

$$= \begin{bmatrix} x - X_1 \\ x - X_2 \end{bmatrix}^T \quad \underline{4} \quad \underline{1}$$

1 2

3 $x_2 + 1$

$+ \; 9($ 1

$-1 \quad \frac{1}{2}$

x_1x_2 1 2 2π

2π 2 27 1 2)

$$\therefore f_{X_1X_2}(x_1, x_2) == \frac{0.19245}{2\pi} e^{-\frac{1}{2}\left[\frac{4}{27}(x_1-2)^2 + \frac{2}{9}(x_1-2)(x_2+1) + \frac{1}{3}(x_2+1)^2\right]}$$

1 1 9 3

27 9

$2 \quad (1-\rho^2)\sigma^2_{x_1} \quad (1-\rho^2)\sigma_{x_1}\sigma_{x_2} \quad (1-\rho^2)\sigma^2_{x_2}$

$\frac{1}{3}$ $x_2 - \overline{X_2}$

$_2 + 1) + {}_3($ $_2$

x

$_2 + 1) + {}_3($ $_2$

$x_2 -$ X_2

$x_1 - X_1$

h __i $\frac{4}{}$ . . .

$\frac{1}{}$ $x_1 - X_1$

$$= \Big[(x_1 - 2)^2 \tfrac{4}{27} + (x_1 - 2)(x_2 + 1)\tfrac{1}{} + (x_1 - 2)(x_2 + 1)\tfrac{1}{} + (x_2 + 1)^2 \tfrac{1}{} \Big]$$

$\overline{27}$ 1 2)

$2(1 - 0.5^2)$ $\overline{\quad}$ 9 3×2 $\overline{\quad}$ 4

10.392π 1.5 $\overline{\quad}$ 9 6 $\overline{\quad}$ 4

= ▪ 29

9 ▪ ▪ ▪

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—

[

$x \;-\; 2)(x$

$_{27} \quad \underline{2}$

$= \;{}^{4}\,(x\;-\;\;{}^{2}$

$\frac{1}{+ 1)} \; x)^2$

9 9 3

$$\therefore f \quad (x\ , x\ ) = \qquad Exp$$

$\overline{2}$] C $]^{-1}$ $[x - X]$ ’

$+ \ _9($ 1

$$= \frac{0.19245}{} Exp \left(- \frac{1}{} \frac{4}{} (x - \right.^{2}$$

$\frac{1}{} x + 1)^2$ i

–

$x - 2)(x$

$\sqrt{1-\rho^2}$

Another Method:

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}} \times Exp\left(-\frac{1}{} \quad (x_1 - X_1)^2 - \frac{2\rho(x_1-X_1)(x_2-X_2)}{} + (x_2-X_2)^2 \right)$$

$$
f_{X_1X_2}(x_1, x_2) = \frac{}{2\pi 3\sqrt{1-0.5^2}}
$$

$$
\times Exp\left(-\frac{1}{}\left[(x_1-2)^2 - \underline{2(-0.5)(x_1-2)(x_2+1)} + (x_2+1)^2\right]\right)
$$

$$
= \frac{1}{} \times Exp\left(-\frac{1}{}\left[(x_1-2)^2 + \underline{(x_1-2)(x_2+1)} + (x_2+1)^2\right]\right)
$$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{10.392\pi} \times Exp\left(-\frac{1}{1.5}\left[\frac{(x_1-2)^2}{9} + \frac{(x_1-2)(x_2+1)}{6} + \frac{(x_2+1)^2}{4}\right]\right)$$

(g) Linear transformation of Joint PDF

$$f_{Y_1Y_2}(y_1,y_2) = \frac{\left|[C_Y]^{-1}\right|^{\frac{1}{2}}}{2\pi}\, Exp\left\{-\frac{1}{2}[y-\bar{Y}]^T\,[C_Y]^{-1}\,[y-\bar{Y}]\right\}$$

where $C_Y = [T][C_X][T]^T$

$$Y_1 = -X_1 + X_2; \qquad Y_2 = -2X_1 - 3X_2$$

$$T = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \qquad [T]^T = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} \qquad \text{and given} \qquad C_X = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}$$

$$C_Y = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix}$$

$$[C_Y]^{-1} = \frac{1}{[19\times 36 - 3\times 3]}\begin{bmatrix} 36 & -3 \\ -3 & 19 \end{bmatrix} = \begin{bmatrix} \frac{36}{675} & \frac{-3}{675} \\ \frac{-3}{675} & \frac{19}{675} \end{bmatrix}$$

$$\left|[C_Y]^{-1}\right| = \frac{36}{675}\times\frac{19}{675} - \frac{-3}{675}\times\frac{-3}{675} = \frac{1}{675}$$

$$\left|[C_Y]^{-1}\right|^{\frac{1}{2}} = \frac{1}{\sqrt{675}} = \frac{1}{25.98}$$

$$f_{Y_1Y_2}(y_1,y_2) = \frac{\frac{1}{25.98}}{2\pi}\, Exp\left\{-\frac{1}{2}[y-\bar{Y}]^T\,[C_Y]^{-1}\,[y-\bar{Y}]\right\}$$

$$[y-\bar{Y}]^T\,[C_Y]^{-1}\,[y-\bar{Y}] = \begin{bmatrix} y_1-\bar{Y}_1 \\ y_2-\bar{Y}_2 \end{bmatrix}^T \left[C_Y\right]^{-1} \begin{bmatrix} y_1-\bar{Y}_1 \\ y_2-\bar{Y}_2 \end{bmatrix}$$

$$= \begin{bmatrix} y_1+3 \\ y_2+1 \end{bmatrix}^T \begin{bmatrix} \frac{36}{675} & \frac{-3}{675} \\ \frac{-3}{675} & \frac{19}{675} \end{bmatrix}\begin{bmatrix} y_1+3 \\ y_2+1 \end{bmatrix}$$

$$= \begin{bmatrix} y_1+3 & y_2+1 \end{bmatrix}\begin{bmatrix} \frac{36}{675} & \frac{-3}{675} \\ \frac{-3}{675} & \frac{19}{675} \end{bmatrix}\begin{bmatrix} y_1+3 \\ y_2+1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36}{675}(y_1+3) + (\frac{-3}{675})(y_2+1) & (\frac{-3}{675})(y_1+3) + \frac{19}{675}(y_2+1) \end{bmatrix}\begin{bmatrix} y_1+3 \\ y_2+1 \end{bmatrix}$$

$$= \left[\frac{36}{675}(y_1+3)^2 - (\frac{3}{675})(y_2+1)(y_1+3) - (\frac{3}{675})(y_1+3)(y_2+1) + \frac{19}{675}(y_2+1)^2\right]$$

$$= \frac{36}{675}(y_1+3)^2 - \frac{6}{675}(y_1+3)(y_2+1) + \frac{19}{675}(y_2+1)^2$$

$$\boxed{f_{Y_1Y_2}(y_1, y_2) = \frac{1}{51.96\pi} Exp\left[\frac{36}{675}(y_1+3)^2 - \frac{6}{675}(y_1+3)(y_2+1) + \frac{19}{675}(y_2+1)^2 \right]}$$

Problem: Two random variables X and Y have mean value $\overline{X} = 1$ and $\overline{Y} = 1$; Variance $\sigma_X^2 = 4, \sigma_X^2 = 2$ and correlation coefficient $\rho_{XY} = 0.2$. Define two random variable $V = -X - Y$, $W = 2X + Y$. Find (a) correlation of V and W (i.e., R_{VW}) (b) Correlation coefficient ρ_{VW},

Solution:

$\overline{X} = 1 \quad \overline{Y} = 1; \quad \sigma_X^2 = 4, \quad \sigma_X^2 = 2 \quad \rho_{XY} = 0.2$

$V = -X - Y;\ W = 2X + Y.$

(a)

$$R_{VW} = E[VW]$$
$$= E\left[(-X - Y)(2X + Y)\right]$$
$$= -2E[X^2] - 2E[XY] - E[Y^2]$$
$$= -2(5) - 2(1.5656) - 3$$
$$= -16.1312$$
$$\boxed{\therefore R_{VW} = -16.1312}$$

$$\sigma_X^2 = E[X^2] - \left(E[X]\right)^2$$
$$E[X^2] = \sigma_X^2 + \left(E[X]\right)^2 = 4 + 1^2 = 5$$
$$E[Y^2] = \sigma_Y^2 + \left(E[Y]\right)^2 = 2 + 1^2 = 3$$
$$E[XY] = C_{XY} + \overline{X}\,\overline{Y}$$
$$= \rho_{XY}\sigma_X\sigma_Y + \overline{X}\,\overline{Y}$$
$$= (0.2)\sqrt{4}\sqrt{2} + 1(1) = 1.5656$$

(b)

$$\rho_{VW} = \frac{C_{VW}}{\sigma_V \sigma_W}$$

$$\sigma_V^2 = E\left[(V - \overline{V})^2\right]$$
$$= E\left[\left((-X - Y) - \overline{(-X - Y)}\right)^2\right]$$
$$= E\left[\left(-(X - \overline{X}) - (Y - \overline{Y})\right)^2\right]$$
$$= E\left[(X - \overline{X})^2 + (Y - \overline{Y})^2 - 2(X - \overline{X})(Y - \overline{Y})\right]$$
$$= \sigma_X^2 + \sigma_Y^2 - C_{XY} \qquad \because C_{XY} = \rho_{XY}\sigma_X\sigma_Y = 0.2\sqrt{4}\sqrt{2} = 0.5656$$
$$= 4 + 2 + 2(0.56568)$$
$$= 7.1313$$

$$\boxed{\therefore \rho_V = \sqrt{7.1313} = 2.6704}$$

$$\sigma_W^2 = E\left[(W - \overline{W})^2\right]$$
$$= E\left[\left((2X + Y) - \overline{(2X + Y)}\right)^2\right]$$

$$= E\left[2(X-\bar{X}) + (Y-\bar{Y})\right]^2$$

$$= E\left[4(X-\bar{X})^2 + (Y-\bar{Y})^2 + 4(X-\bar{X})(Y-\bar{Y})\right]$$

$$= 4\sigma_X^2 + \sigma_Y^2 + 4C_{XY} \qquad \because\ C_{XY} = \rho_{XY}\,\sigma_X\sigma_Y = 0.2\sqrt{4}\sqrt{2} = 0.5656$$

$$= 4(4) + 2 + 4(0.56568)$$

$$= 20.2624$$

$$\boxed{\therefore\ \rho_W = \sqrt{20.2624} = 4.5013}$$

$$R_{VW} = C_{VW} + \bar{V}\,\bar{W}$$

$$\bar{V} = \overline{-X-Y} = -\bar{X} - \bar{X} = -1 - 1 = -2$$

$$\bar{W} = \overline{2X+Y} = 2\bar{X} + \bar{X} = 2(1) + 1 = 3$$

$$C_{VW} = R_{VW} - \bar{V}\,\bar{V}$$

$$= -16.1312 - (-2)(3) = -10.1312$$

$$\boxed{\therefore\ C_{VW} = -10.1312}$$

$$\rho_{VW} = \frac{C_{VW}}{\sigma_V\,\sigma_W} = \frac{-10.1312}{2.6704 \times 4.5013} = -0.84284$$

$$\boxed{\rho_{VW} = -0.84284}$$

6.9 Transformation of multiple random variables

Let $X_1, X_2, \ldots X_N$ are 'N' input random variable and transformed to $Y_1, Y_2, \ldots Y_N$ i.e.,

$$Y_i = T_i\left[X_1, X_2 \ldots X_N\right]; \qquad i = 1, 2, \ldots N$$

$$X_i = T_j^{-1}\left[X_1, X_2 \ldots X_N\right]; \qquad j = 1, 2, \ldots N$$

the relation between PDFs

$$f_{Y_1 Y_2 \ldots Y_N}(y_1, y_2, \ldots y_N) = f_{X_1 X_2 \ldots X_N}(x_1, x_2, \ldots x_N) \cdot |J|$$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} & \ldots & \frac{\partial X_1}{\partial Y_N} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_2}{\partial Y_3} & \ldots & \frac{\partial X_2}{\partial Y_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_N}{\partial Y_1} & \frac{\partial X_N}{\partial Y_2} & \frac{\partial X_N}{\partial Y_3} & \ldots & \frac{\partial X_N}{\partial Y_N} \end{vmatrix} \qquad \text{Note: } f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$\overline{X_1}$ $\overline{}_2$

Problem: Two random variables X_1 and X_2 are defined by $X_1 = 0, X_2 = 1,$ $= 4$ and $R_{X_1X_2}$

$^2 = 2, X^2$ $= -2$. The two new random variables Y_1 and Y_2 are

$Y_1 = 2X_{!} + X_2, Y_2 = -X_1 - 3X_2$. Find $\overline{Y_1}, \overline{Y_2}, \overline{Y_1^2}, \overline{Y_2^2}, R_{Y_1Y_2}, \sigma^2_{X_1}, \sigma^2_{X_2}$ and $f_{Y_1Y_2}(y_1, y_2)$?

Solution: Given data

$\overline{X_1} = E[X_1] = 0$ $\quad$ $\overline{X_2} = E[X_2] = -1$

$E[X_1^2] = \overline{X_1^2} = 2$ $\quad$ $E[X_2^2] = \overline{X_2^2} = 4$

$R_{X_1X_2} = E[X_1X_2] = -2$

$Y_1 = 2X_1 + X_2$ $\quad$ $Y_2 = -X_1 - 3X_2$

(i) $\overline{Y_1} = E[Y_1]$

$= E[2X_1 + X_2]$

$= 2E[X_1] + E[X_2]$

$= 2(0) + (-1)$

$= -1$

(ii) $\overline{Y_2} = E[Y_2]$

$= E[-X_1 - 3X_2]$

$= -E[X_1] - 3E[X_2]$

$= (0) - 3(-1)$

$= 3$

(iii) $\overline{Y_1^2} = E[Y_1^2]$

$= E[(2X_1 + X_2)^2]$

$= E[4X_1^2 + X_2^2 + 4X_1X_2]$

$= 4E[X_1^2] + E[X_2^2] + 4E[X_1X_2]$

$= 4(2) + 4 + 4(-2)$

$= 4$

(iv) $\overline{Y_2^2} = E[Y_2^2]$

$= E[(-X_1 - 3X_2)^2]$

$= E[X_1^2 + 9X_2^2 + 6X_1X_2]$

$= E[X_1^2] + 9E[X_2^2] + 6E[X_1X_2]$

$= 2 + 9(4) + 6(-2)$

$= 32$

(v) $R_{Y_1Y_2} = E[Y_1Y_2] = E[Y_1]E[Y_2] = (-1)(3) = -3$

(vi) Given

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$= \frac{1}{-6+1} \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\boxed{X_1 = \frac{3}{5}Y_1 + \frac{1}{5}Y_2; \qquad X_2 = -\frac{1}{5}Y_1 - \frac{2}{5}Y_2}$$

$$E[X_1] = \frac{3}{5}E[Y_1] + \frac{1}{5}E[Y_2] = \frac{3}{5}(-1) + \frac{1}{5}(3) = -\frac{3}{5} + \frac{3}{5} = 0$$

$$E[X_2] = -\frac{1}{5}E[Y_1] - \frac{2}{5}E[Y_2] = -\frac{1}{5}(-1) - \frac{2}{5}(3) = \frac{1}{5} - \frac{6}{5} = -1$$

(vi) $\sigma^2_{X_1} = E[X_1^2] - (E[X_1])^2 = 2 - 0^2 = 2$

(vii) $\sigma^2_{X_2} = E[X_2^2] - (E[X_2])^2 = 4 - (-1)^2 = 3$

(viii) $f_{Y_1Y_2}(y_1, y_2) = ?$

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(x_1, x_2)\cdot|J|$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{vmatrix} = \frac{3}{5}\left(-\frac{2}{5}\right) - \frac{1}{5}\left(-\frac{1}{5}\right) = -\frac{6}{25} + \frac{1}{25} = -\frac{1}{5}$$

$$\boxed{\therefore f_{Y_1Y_2}(y_1, y_2) = \left|-\frac{1}{5}\right| f_{X_1X_2}\left(\frac{3Y_1}{5} + \frac{Y_2}{5}; \; -\frac{Y_1}{5} - \frac{2Y_2}{5}\right)}$$

CHAPTER 7

Random Process

7.1 Random Process Concept

The concept of a random process is based on enlarging the random variable concept to include time. The random variable is function of sample coins or sample space and the random process is both sample space and time then it is called random process or stochastic process and it is defined as $X(t, s)$.

The random process $X(t, s)$ has family of specific values $x(t, s)$. A random process is sort form can be represented as $X(t)$, it has family of specific values $x(t)$.

Random process can be represented in three methods.

1. Both time 't' and sample space'S' are variable.
2. Time 't' is fixed and sample space 'S'is variable.
3. Time 't' is fixed and sample space 'S'is fixed.

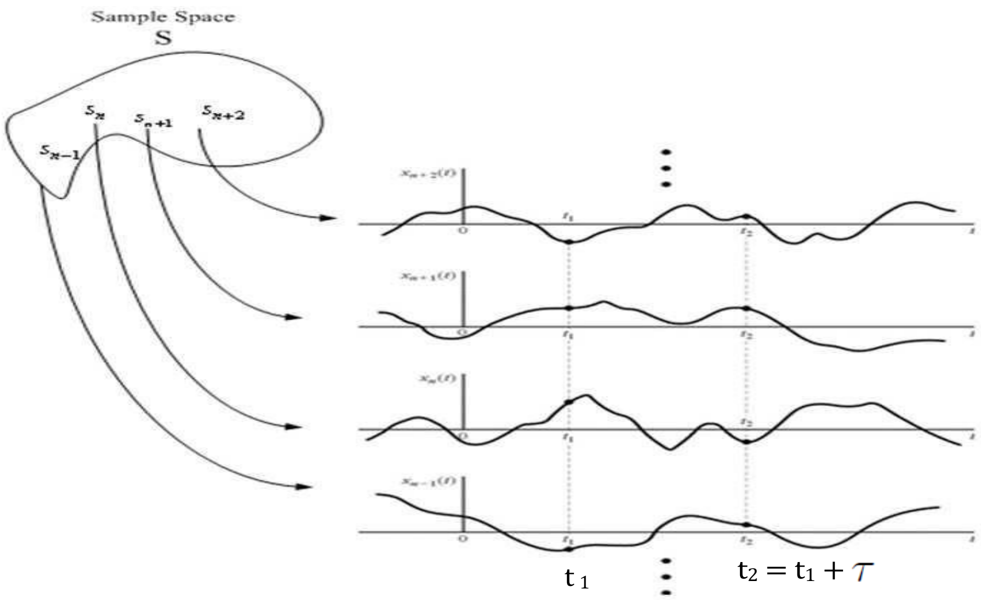


Fig. 7.1 Example: Random process

Exampe: Let us consider an experiment of measuring the temperature of a room with different or collection of room temperature using thermometer. Each thermometer is a random variable which can take on any value from the sample space 'S'. Also at different times the reading of thermometers may be different.Thus the room temperature is a function of a both the sample space and time. In this example, the concept of random variable can extended by taking into consideration of time dimension. Here we assign a time function $x(t, s)$ to every outcome 's'. There will be a family of all such functions. The family $X(t, S)$ is known as "random process" or "stochastic process". In place of $x(t, s)$ and $X(t, S)$, the sort form notation $x(t)$ and $X(t)$ are often used.

The Fig. 7.1 shows random process, 'S' is sample space with sample S_1, S_2, S_3. Sample S_1 is corresponds to thermometer1 readings i.e., $x_1(t)$. S_2 and S_3 are corresponds to thermometer2 and thermometer3 readings respectively.

To determine the statistics of the room temperature, say mean value two methods are used.

7.1.1 Time 't' is fixed

The random variable corresponding to random process can be obtained by fixing time $T = t_1, t_2, t_3, \dots t_N$ The random variable X_1 is obtained at fixing time $t = t_1$, then

$$X(t)\Big|_{t=t_1} = X_1 = \{A_1, A_2, A_3\} \qquad similarly(= X(t_1))$$

The random variable X_2 is obtained at fixing time $t = t_2$, then

$$X(t)\Big|_{t=t_2} = X_2 = \{B_1, B_2, B_3\} \qquad similarly(= X(t_2))$$

then the PDF of a random variable X_1 and X_2 can be obtained by calculating probability of a random variable.

Let $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are represents the PDF's of random variable X_1 and X_2. The CDF's can be obtained by integrating or adding the PDF's $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ are represents the CDF's of random variable's X_1 and X_2.

∴ The statistical parameters of random process is mean value or expectation or statistical average or ensemble average $E[X_k]$.

$$E[X_k] = \int_{x_k=-\infty}^{\infty} x_k f_{X_k}(x_k) dx_k$$

7.1.2 Time averages or entire time scale

The mean vale may be calculated over the entire time scale or time averages.

$$A[x_1(t)] = \langle x_1(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_1(t)dt$$

This is called "Time average". Similarly mean values of $x_2(t)$ and $x_3(t)$ can be calculated.

∴ Total Time average $A[X(t)] = \langle x_1(t) \rangle = A[x_1(t), x_2(t), x_3(t)$.

Correlation of random process:

The random process $X(t)$ is expected value of random variable X_1 and X_2 is

$$\begin{aligned} E[X(t_1).X(t_2)] &= R_{X_1X_2}(t_1, t_2) \\ &= E[X(t_1).X(t+\tau)] \\ &= R_{X_1X_2} \end{aligned}$$

Time averaging Correlation:

$$\begin{aligned} A[x_1(t).x_2(t+\tau)] &= \lim_{T\to\infty} \frac{1}{T} \int_0^{\infty} x_1(t).x_2(t+\tau)dt \\ &= R_{X_1X_2}(\tau) \end{aligned}$$

7.2 Classifications of Random Process

1. Non-Deterministic process
   - i. Continuous random process
   - ii. Discrete random process
   - iii. Continuous random sequence or Continuous sequence random process
   - iv. Discrete random sequence or Discrete sequence random process
2. Deterministic random process
3. Stationary random process
   - i. First order Stationary random process
   - ii. Second order Stationary random process
   - iii. N^{th} order Stationary random process

iv. Strict sense stationary random process (SSS)

v. Wide sense stationary random process (WSS)

4. Non-Stationary random process

5. Ergodic random process

1. **Non-Deterministic process:** If the future values of any sample function can not be predicted exactly from observed past values, the process is called "Non-Deterministic process".

   i. Continuous random process: If the future values are not predicted in advance and values are continuously varying with respect to time then it is called "continuous random process". Examples are

      – Temperature measured using thermometer.

      – Thermal noise generated by resistor.

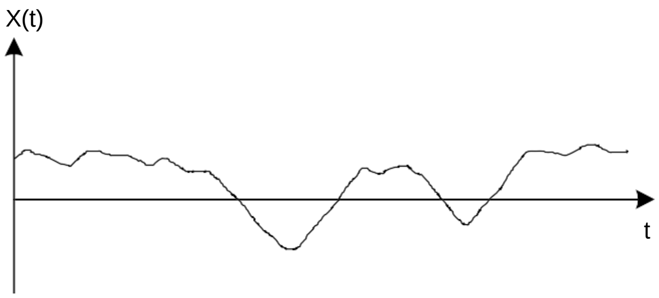


Fig. 7.2

   ii. Discrete random process: If $X(t)$ is discrete with respect to time 't' then random process is called "Discrete random process". It has only two set of values. Ex: Logic '1' and '0' generated by personal computer.

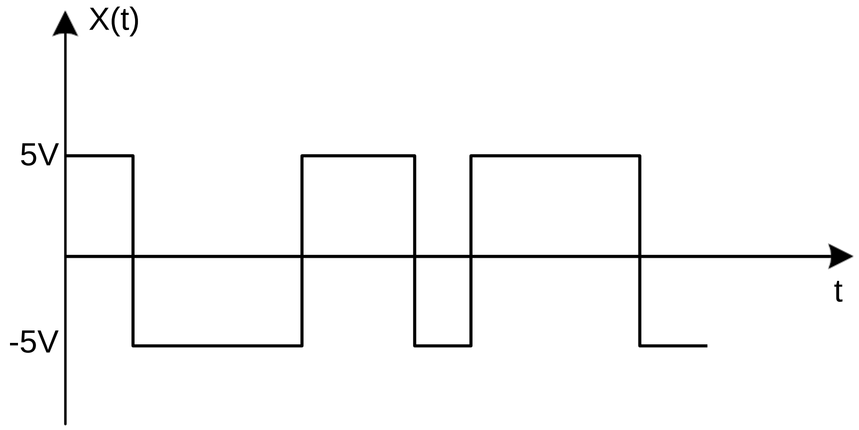


Fig. 7.3

iii. Continuous sequence random process: A random process for which $X(t)$ is continuous but time has only discrete values is called a "continuous sequence random process". This can be obtained by sampling continues random process.

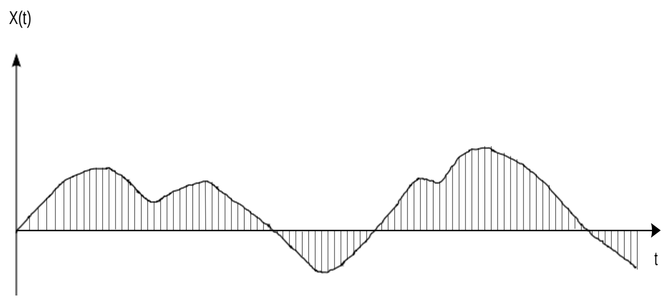


Fig. 7.4

iv. Discrete sequence random process: A random process for which $X(t)$ and 't' are discrete is called a "discrete sequence random process". This can be obtained by sampling discrete random process.

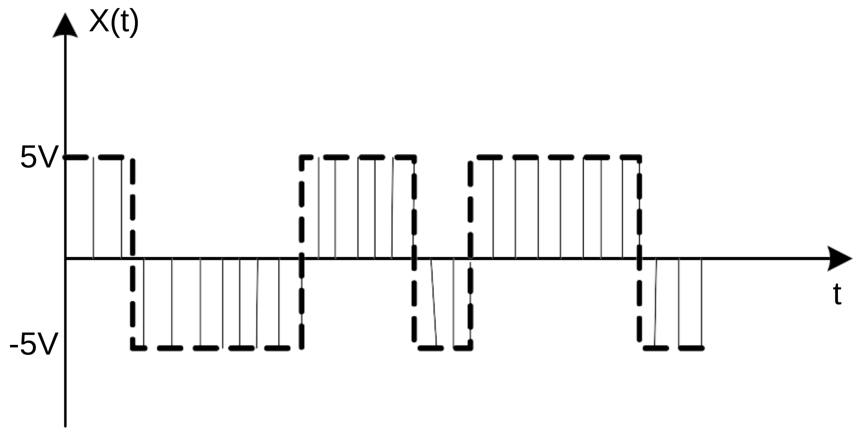


Fig. 7.5

2. **Deterministic random process:** If the future values of any sample function can be predicted exactly from observed past values, the process is called "Deterministic process".

   • Example: $X(t) = A\cos(\omega_0 t + \vartheta) = A\cos(2\pi f_0 t + \vartheta)$
   Here $A, f_0,$ or ω_0 and ϑ are random variable.

3. **Stationary random process:** If the statistical parameters of a random process are constant with respect to time then it is called s"stationary random process". This means the random process $X(t)$ and $X(t + \tau)$ posses the same statistical properties for any value of τ (i.e., not affected by a shift in the time).

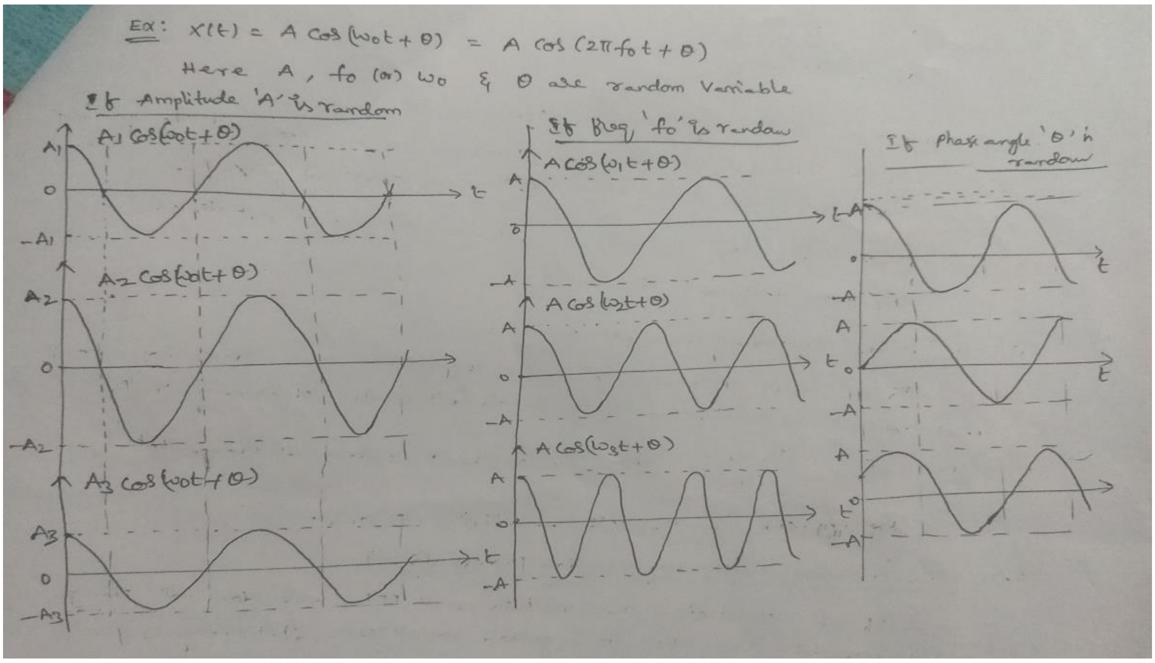


Fig. 7.6

The physical meaning of stationary is that a time translation of a sample function results in another sample function of the random process having the same probability.

i. First order Stationary random process: If the first order PDF and expectation constant doesn't change with respect to time, then the random process is called "first order Stationary random process". Ex:
   - $f_{X_1}(x_1)$ is constant. i.e., does not with respect to time.
   - $E[X_1]$ is constant.

ii. Second order Stationary random process: If the second order PDF and expectation constant doesn't change with respect to time, then the random process is called "2^{nd} order Stationary random process". Ex:
   - $f_{X_1 X_2 \ldots X_N}(x_1, x_2 \ldots x_N)$ is constant. i.e., does not with respect to time.
   - $E[X(t_1)X(t_2) \ldots X(t_N)]$ is constant.

iii. N^{th} order Stationary random process: If the N^{th} order PDF and expectation constant doesn't change with respect to time, then the it is called "N^{th} order Stationary random process". Ex:
   - $f_{X_1 X_2}(x_1, x_2)$ is constant. i.e., does not with respect to time.
   - $E[X(t_1)X(t_1 + \tau)]$ is constant.

iv. Strict sense stationary random process (SSS): If all statistical parameters and PDF's are does not change with respect to time then it is called strict sense stationary random process.

v. Wide sense stationary random process (WSS): If expectation or mean is constant and correlation is function of $\tau = t_2 - t_1$ then it is called wide sense stationary random process.

   – $E[X(t)]$ is constant.
   – $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ is constant. or
   – $E[X(t_1)X(t_2)] = E[X(t_1)X(t_1+\tau)]$ is constant.

4. **Non-Stationary random process:** If any statistical parameters is changes with respect to time then it is called "non-stationary random process".

5. **Ergodic random process:** If statistical averages are equal to time averages then it is called "ergodic random process".

   • Mean Ergodic r.p: $E[X(t)] = A[X(t)]$
   • Correlation Ergodic r.p: $E[X(t)X(t+\tau)] = A[X(t)X(t+\tau)]$
   • Variance Ergodic r.p: $E[(X(t)X(t+\tau))^2] = A[(X(t)X(t+\tau))^2]$

Problem 1: A random process $X(t) = A\cos(\omega_0 t + \vartheta)$ is stationary if A and ω_0 are constants and ϑ is a uniformly distributed variable on the interval $(0, 2\pi)$. Show that it is WSS r.p.

Solution: Given $X(t) = A\cos(\omega_0 t + \vartheta)$; where A and ω_0 are constants and $\vartheta \longrightarrow (0, 2\pi)$; $f_\vartheta(\vartheta) = \frac{1}{2\pi}$. The ϑ is uniformly distributed between 0 to 2π. The distribution shown in Fig. 7.12.

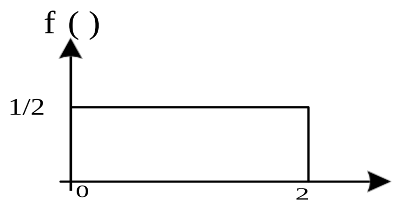


Fig. 7.7

If mean value and auto correlation function of r.p is a function of time, 't' then it is not stationary.

1. Expectation or mean value:

$$E[X(t)] = \overline{X(t)} = \int_{\vartheta=0}^{2\pi} x(t) f_\vartheta(\vartheta) d\vartheta$$
$$= \int_{\vartheta=0}^{2\pi} A\cos(\omega_0 t + \vartheta).\frac{1}{2\pi} d\vartheta$$
$$= \frac{A}{2\pi}\int_{\vartheta=0}^{2\pi} \cos(\omega_0 t + \vartheta) d\vartheta$$
$$= \frac{A}{2\pi}[\sin(\omega_0 t + \vartheta)]_0^{2\pi}$$
$$= \frac{A}{2\pi}[\sin(\omega_0 t + 2\pi) - \sin\omega_0 t]$$
$$= \frac{A}{2\pi}[\sin\omega_0 t - \sin\omega_0 t]$$
$$= 0$$

∴ $E[X(t)] = \overline{X(t)} = 0$. It is constant.

2. Correlation:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$= \int_{\vartheta=0}^{2\pi} x(t)X(t+\tau) f_\vartheta(\vartheta) d\vartheta$$
$$= \int_{\vartheta=0}^{2\pi} A\cos(\omega_0 t + \vartheta).A\cos(\omega_0(t+\tau) + \vartheta)\frac{1}{2\pi} d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} 2\cos(\omega_0 t + \vartheta).\cos(\omega_0 t + \omega_0\tau + \vartheta) d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} \cos(\omega_0 t + \vartheta + \omega_0 t + \omega_0\tau + \vartheta) + \cos(\omega_0 t + \vartheta - \omega_0 t - \omega_0\tau - \vartheta) d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} \cos(2\omega_0 t + 2\vartheta + \omega_0\tau) + \cos(\omega_0\tau) d\vartheta$$
$$= \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + 2\vartheta + \omega_0\tau)}{2}\right]_0^{2\pi} + \frac{A^2}{4\pi}[\cos(\omega_0\tau)]\left[\vartheta\right]_0^{2\pi}$$
$$= \frac{A^2}{4\pi}[\cos(\omega_0\tau)]\left[\vartheta\right]_0^{2\pi} + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + 4\pi + \omega_0\tau) - \sin(2\omega_0 t + \omega_0\tau)}{2}\right]$$
$$= \frac{A^2}{4\pi}\cos(\omega_0\tau)[2\pi - 0] + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + \omega_0\tau) - \sin(2\omega_0 t + \omega_0\tau)}{2}\right]$$
$$= \frac{A^2}{2}\cos\omega_0\tau + 0$$
$$= \frac{A^2}{2}\cos\omega_0\tau$$

∴$R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$. This solution does not contain variable 't'.
So, both $E[X(t)]$ and $E[X(t)X(t+\tau)]$ are constant, then it is WSS.

Problem 2: A random process $X(t) = A\cos(\omega_0 t + \vartheta)$ is not stationary if A and ω_0 are constants and ϑ is a uniformly distributed variable on the interval $(0, \pi)$. Show that it is not WSS r.p.

Solution: Given $X(t) = A\cos(\omega_0 t + \vartheta)$; where A and ω_0 are constants and $\vartheta \longrightarrow (0, \pi)$; $f_\vartheta(\vartheta) = \frac{1}{\pi}$ The ϑ is uniformly distributed between 0 to π. The distribution shown in Fig. 7.8.

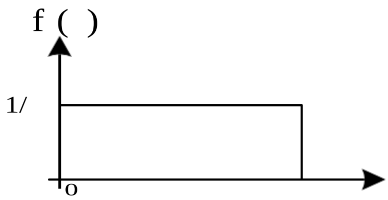


Fig. 7.8

If mean value and auto correlation function of r.p is a function of time, 't' then it is not stationary.

1. Expectation or mean value:

$$E[X(t)] = \overline{X(t)} = \int_{\vartheta=0}^{\pi} x(t) f_\vartheta(\vartheta)\, d\vartheta$$

$$= \int_{\vartheta=0}^{\pi} A\cos(\omega_0 t + \vartheta).\frac{1}{\pi} d\vartheta$$

$$= \frac{A}{\pi}\int_{\vartheta=0}^{2\pi} \cos(\omega_0 t + \vartheta)\, d\vartheta$$

$$= \frac{A}{\pi}\Big[\sin(\omega_0 t + \vartheta)\Big]_0^{\pi}$$

$$= \frac{A}{\pi}\Big[\sin(\omega_0 t + \pi) - \sin\omega_0 t\Big]$$

$$= \frac{A}{\pi}\Big[-\sin\omega_0 t - \sin\omega_0 t\Big]$$

$$= \frac{-2A}{\pi}\sin\omega_0 t$$

$$\therefore\ E[X(t)] = \frac{-2A}{\pi}\sin\omega_0 t.$$

It is not constant. i.e., varying with 't'. So, it is not stationary r.p.

2. Correlation:

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

$$= \int_{\vartheta=0}^{\pi} x(t)X(t+\tau)f_{\vartheta}(\vartheta)d\vartheta$$

$$= \int_{\vartheta=0}^{\pi} A\cos(\omega_0 t+\vartheta).A\cos(\omega_0(t+\tau)+\vartheta)\frac{1}{\pi}d\vartheta$$

$$= \frac{A^2}{2}\int_{\vartheta=0}^{\pi} \cos(2\omega_0 t+2\vartheta+\omega_0\tau)+\cos(\omega_0\tau)d\vartheta$$

$$= \frac{A^2}{2\pi}\left[\frac{\sin((2\omega_0 t+2\vartheta+\omega_0\tau)}{2}\right]_0^{\pi} + \frac{A^2}{2\pi}\cos(\omega_0\tau)\left[\vartheta\right]_0^{\pi}$$

$$= \frac{A^2}{2\pi}\left[\frac{\sin((2\omega_0 t+2\pi+\omega_0\tau)-\sin((2\omega_0 t+\omega_0\tau)}{2}\right] + \frac{A^2}{2\pi}\cos(\omega_0\tau)\left[\pi-0\right]$$

$$= \frac{A^2}{2\pi}\left[\frac{\sin((2\omega_0 t+\omega_0\tau)-\sin((2\omega_0 t+\omega_0\tau)}{2}\right] + \frac{A^2}{2\pi}\cos(\omega_0\tau)\left[\pi-0\right]$$

$$= 0 + \frac{\pi A^2}{2\pi}\cos(\omega_0\tau)$$

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$

$$\therefore R_{XX}(\tau) = \frac{A^2}{2}\cos(\omega_0\tau).$$

This solution does not contain variable 't'. So it is constant.

$\therefore$ $E[X(t)]$ is 't' varying function and $E[X(t)X(t+\tau)]$ is constant. So, it is not WSS r.p.

Problem 3: A random process $X(t) = A\cos(\omega_0 t+\vartheta)$ where ϑ and ω_0 are constants and Amplitude (A) is a random variable from $-a$ to a (i.e, uniformly distributed). Find Expectation and Auto correlation?

Solution: Given $X(t) = A\cos(\omega_0 t+\vartheta)$; where ϑ and ω_0 are constants and $A \rightarrow (-a, a)$; $f_A(A) = \frac{1}{2a}$. The amplitude A is uniformly distributed between $-a$ to a. The distribution shown in Fig. 7.9.

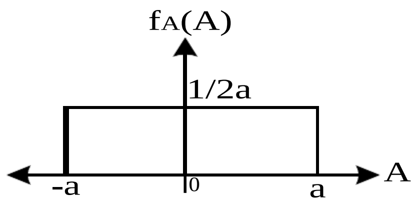


Fig. 7.9

1. Expectation or mean value $E[X(t)]$:

$$E[X(t)] = \overline{X(t)} = \int_{A=-a}^{a} x(t) f_A(A)\, dA$$
$$= \int_{A=-a}^{a} A \cos(\omega_0 t + \vartheta).\frac{1}{2a} dA$$
$$= \frac{\cos(\omega_0 t + \vartheta)}{2a} \int_{A=-a}^{a} A\, dA$$
$$= \frac{\cos(\omega_0 t + \vartheta)}{2a} \left[\frac{A^2}{2}\right]_{-a}^{a}$$
$$= \frac{\cos(\omega_0 t + \vartheta)}{2a} \left[\frac{a^2}{2} - \frac{(-a)^2}{2}\right]$$
$$= 0$$

∴ $E[X(t)] = 0$. It is constant. So, it is stationary r.p.

2. Auto Correlation $(R_{XX}(\tau)$):

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$= E[A\cos(\omega_0 t + \vartheta).A\cos(\omega_0 t + \omega_0 \tau + \vartheta)]$$
$$= E\left[\frac{A^2}{2}\left(\cos(\omega_0 \tau) + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)\right)\right]$$
$$= \frac{\cos \omega_0 \tau}{2} E[A^2] + \frac{\cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)}{2} E[A^2]$$
$$= \left[\frac{\cos \omega_0 \tau + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)}{2}\right] \times E[A^2]$$
$$= \left[\frac{\cos \omega_0 \tau + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)}{2}\right] \times \int_{A=-a}^{a} A^2.\frac{1}{2a} dA$$
$$= \left[\frac{\cos \omega_0 \tau + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)}{2}\right] \times \frac{1}{2a}\left[\frac{A^3}{3}\right]_{-a}^{a}$$
$$= \left[\frac{\cos \omega_0 \tau + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)}{2}\right] \times \frac{1}{2a}\left[\frac{2a^3}{3}\right]$$
$$= \frac{a^2}{6}\left[\cos \omega_0 \tau + \cos(2\omega_0 t + 2\vartheta + \omega_0 \tau)\right]$$

Problem 4: A random process $X(t) = A\cos(\omega_0 t + \vartheta)$ where ϑ and A are constants and frequency ω_0 is a uniform random variable from 0 to 100 *rad/sec*. Find Expectation and Auto correlation?

Solution: Given $X(t) = A\cos(\omega_0 t + \vartheta)$; where A and ϑ are constants and $\omega_0 \rightarrow (0, 100)$; $f_A(A) = \frac{1}{100}$ The amplitude ω_0 is uniformly distributed between 0 to 100. The distribution shown in Fig. 7.10.

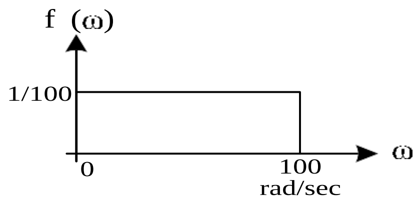


Fig. 7.10

1. Expectation or mean value $E[X(t)]$:

$$E[X(t)] = \int_{\omega_0=0}^{100} x(t) f_{\omega_0}(\omega_0) d\omega_0$$
$$= \int_{\omega_0=0}^{100} A\cos(\omega_0 t + \vartheta).\frac{1}{100} d\omega_0$$
$$= \frac{A}{100}\int_{\omega_0=0}^{100} \cos(\omega_0 t + \vartheta) d\omega_0$$
$$= \frac{A}{100}\left[\frac{\sin(\omega_0 t + \vartheta)}{t}\right]_0^{100}$$
$$= \frac{A}{100t}\left[\sin(100t + \vartheta) + \sin(\vartheta)\right]$$

$\therefore E[X(t)] = \frac{A}{100t}\left[\sin(100t + \vartheta) + \sin(\vartheta)\right]$. It is consists parameter 't'. So, it is not stationary r.p.

2. Auto Correlation ($R_{XX}(\tau)$):

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$= E[A\cos(\omega_0 t + \vartheta).A\cos(\omega_0 t + \omega_0\tau + \vartheta)]$$
$$= E\left[\frac{A^2}{2}\cos(\omega_0\tau) + \cos(2\omega_0 t + 2\vartheta + \omega_0\tau)\right]$$
$$= \frac{A^2}{2}\left[E[\cos(\omega_0\tau)] + E[\cos(2\omega_0 t + 2\vartheta + \omega_0\tau)]\right]$$
$$= \frac{A^2}{2}\left[\int_{\omega_0=0}^{100}\cos\omega_0\tau.\frac{1}{100}d\omega_0 + \frac{1}{100}\int_{\omega_0=0}^{100}\cos(2\omega_0 t + \omega_0\tau + 2\vartheta)d\omega_0\right]$$
$$= \frac{A^2}{2}\left[\frac{1}{100}\left[\frac{\sin\omega_0\tau}{\tau}\right]_0^{100} + \frac{1}{100}\left[\frac{\sin(2\omega_0 t + \omega_0\tau + 2\vartheta)}{2t+\tau}\right]_{\omega_0=0}^{100}\right]$$
$$= \frac{A^2}{2}\left[\frac{\sin 100\tau}{100\tau} + \frac{\sin(200t + 100\tau + 2\vartheta)}{200t + 100\tau} - \frac{\sin(2\vartheta)}{200t + 100\tau}\right]$$

Problem 5: A random process $X(t) = K$ where K is uniformly distribution from -1 to 1. Find Expectation and correlation of r.p?

Solution: Given $X(t) = K$; where K is r.v. and $K \to (-1, 1)$; $f_X(K) = \frac{1}{2}$. The amplitude K is uniformly distributed between -1 to 1. The distribution shown in Fig. 7.11.

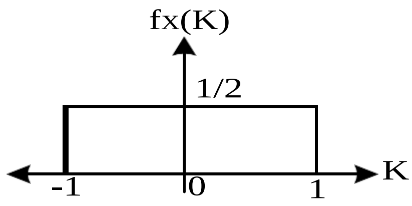


Fig. 7.11

1. Expectation or mean value $E[X(t)]$:

$$E[X(t)] = \int_{K=-1}^{1} x(t) f_X(K)\,dK$$
$$= \int_{K=-1}^{1} K.\frac{1}{2}dK = \frac{1}{2}\Big[\frac{K^2}{2}\Big]_{-1}^{1} = \frac{1}{2}\Big[\frac{1}{2} - \frac{1}{2}\Big] = 0 \qquad \textit{Constant.}$$

2. Correlation ($R_{XX}(\tau)$):

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$= \int_{K=-1}^{1} x(t)x(t+\tau) f_X(K)\,dK$$
$$= \int_{K=-1}^{1} (K)(K)(\frac{1}{2}dK = \frac{1}{2}\Big[\frac{K^3}{3}\Big]_{-1}^{1}$$
$$= \frac{1}{2}\Big[\frac{1}{3} + \frac{1}{3}\Big] = \frac{1}{3} \qquad \textit{Constant.}$$

Both $E[X(t)]$ and $R_{XX}(\tau)$ are constant. So, it is WSS r.p.

Problem 6: A random process $X(t) = aX + b$ where X is constant and a is uniformly distributed from -2 to 2. Find Expectation and correlation of r.p?

Solution: Given $X(t) = aX + b$; where X is constant and a is a r.v. and $a \rightarrow (-2, 2)$; $f_X(a) = \frac{1}{4}$. The variable a is uniformly distributed between -2 to 2. The distribution shown in Fig. **??**.

7.3 Correlation function

Correlation finds the similarities between two random variables in the random process.

7.3.1 Auto-correlation function

Let $X(t)$ be the random process which contain $X(t_1)$ and $X(t_2)$ are random variables. Auto correlation function is defined as

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t_1)X(t+\tau)]$$
$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

7.3.2 Properties of Auto-correlation

1. Mean square or total power of random process can be obtained at $\tau = 0$. i.e.,

$$E[X^2(t)] = R_{XX}(0)$$

Proof. $R_{XX}(\tau) = E[X(t)X(t+\tau)]$
If $\tau = 0$;

$$\begin{aligned} R_{XX}(0) &= E[X(t)X(t)] \\ &= E[X^2(t)] \\ &= \overline{X^2} \end{aligned}$$

□

2. Auto-correlation function is even function $R_{XX}(\tau) = R_{XX}(-\tau)$

Proof. $R_{XX}(\tau) = E[X(t)X(t+\tau)]$
Let $\tau = -\tau$;

$$R_{XX}(-\tau) = E[X(t)X(t-\tau)]$$

Let $u = t - \tau \implies t = u + \tau$

$$\begin{aligned} R_{XX}(-\tau) &= E[X(u+\tau)X(u)] \\ &= R_{XX}(\tau). \end{aligned}$$

□

3. Auto-correlation function has maximum value at origin. $|R_{XX}(\tau)| \leq R_{XX}(0)$

Proof. Consider positive quantity,

$$\left[(X(t_1)+X(t_2))^2\right] \geq 0$$

$$E\left[(X(t_1)+X(t_2))^2\right] \geq 0$$

$$E\left[X^2(t_1)+X^2(t_2)+2X(t_1)X(t_2)\right] \geq 0$$

$$R_{XX}(0) + R_{XX}(0) + 2R_{XX}[(t_1, t_2)] \geq 0 \qquad \because (Property(1) \qquad t_2 = t_1 + \tau)$$

$$R_{XX}(0) + R_{XX}(\tau) \geq 0$$

$$\therefore R_{XX}(\tau) \leq R_{XX}(0)$$

□

4. If $X(t)$ is is independent, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

or If $X(t)$ is ergodic, zero mean and has no periodic component, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

5. If $X(t)$ is periodic then $R_{XX}(\tau)$ will be a periodic with a same period.

6. If a random process with a zero mean has DC component 'A'; $Y(t) = A + X(t)$ then $R_{YY}(\tau) = A^2 + R_{XX}(\tau)$

Proof.

$$Y(t) = A + X(t)$$

$$\begin{aligned}
R_{YY}(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E[(A+X(t))(A+X(t+\tau))] \\
&= E\left[A^2 + AX(t+\tau) + AX(t) + X(t)X(t+\tau)\right] \\
&= E[A^2] + A\underbrace{E[X(t+\tau)]}_{=0} + A\underbrace{E[X(t)]}_{=0} + E[X(t)X(t+\tau)] \\
&= A^2 + R_{XX}(\tau) \qquad \because (mean = 0)
\end{aligned}$$

□

7. If the random process $Z(t)$ is a sum of two random process $X(t)$ and $Y(t)$ that is $Z(t) = X(t) + Y(t)$ then $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$.

Proof.

$$Z(t) = X(t) + Y(t)$$

$$\begin{aligned}
R_{ZZ}(\tau) &= E[Z(t)Z(t+\tau)] \\
&= E\Big[\big(X(t) + Y(t)\big)\big(X(t+\tau) + Y(t+\tau)\big)\Big] \\
&= E\Big[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)\Big] \\
&= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)
\end{aligned}$$

□

8. $R_XX(\tau)$ can not have an arbitrary shape.

9. The auto-correlation of a r.p $X(t)$ is a finite every function.

10. If $E[X(t)] = \overline{X} \neq 0$ and $X(t)$ is ergodic with no periodic components then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$$

Notes:

- Mean of $X(t)$ = DC component
- $E[X^2(t)]$ = Total Power
- $(E[X(t)])^2$ = DC Power
- Variance (σ_X^2) = AC Power
- Standard deviation (σ_X) = rms value

Problem 1: Auto correlation function $R_{XX}(\tau) = 25 + \frac{4}{1+\tau^2}$; is a stationary ergodic process with no periodic components then find mean value and variance?

Solution:

1. Mean square value $E[X^2(t)]$: From Property (1)

$$E[X^2(t)] = R_{XX}(0) = 25 + \frac{4}{1+0} = 29$$

2. Square of Mean value :$(E[X(t)])^2$:

$$E[X(t)] = \overline{X(t)} \neq 0 \quad then$$

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2 = \lim_{|\tau|\to\infty} 25 + \frac{4}{1+\tau^2}$$

$$= \lim_{|\tau|\to\infty} 25 + \frac{\frac{6}{\infty}}{\frac{1}{\tau^2}+6} = 25 + \frac{0}{0+6} = 25$$

$\therefore \overline{X} = \pm 5$

3. Variance: $\sigma_X^2 = m_2 - m_1^2 = E[X^2] - (E[X])^2 = 29 - 5^2 = 4$

Problem 2: The auto correlation function of a WSS r.p is given by $R_{XX}(\tau) = \dfrac{4\tau^2 + 100}{\tau^2 + 4}$. Find mean and variance?

Solution:

1. Mean Square Value: $\overline{X^2} = E[X^2(t)] = R_{XX}(0) = \dfrac{4(0) + 100}{0 + 4} = \dfrac{100}{4} = 25$

2. Mean Value: $(\overline{X})^2 = \lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{|\tau|\to\infty} \dfrac{4 + \frac{100}{\tau^2}}{1 + \frac{4}{\tau^2}} = 4 \therefore \overline{X} = \pm 2$

3. Variance: $\sigma_X^2 = E[X^2(t)] - (E[X])^2 = 25 - 4 = 21$

Problem 3: Assume that an Ergodic random process $X(t)$ has an auto correlation function $R_{XX}(\tau) = 18 + \dfrac{2}{6 + t^2}\left[1 + 4\cos(12\tau)\right]$.

a) Find $|\overline{X}|$

b) Does this process have periodic components?

c) What is the average power in $X(t)$

Solution:

a)

$$
\begin{aligned}
E[X(t)]^2 = (\overline{X})^2 &= \lim_{\tau\to\infty} R_{XX}(\tau) \\
&= \lim_{\tau\to\infty} 18 + \frac{2}{6 + t^2}\left[1 + 4\cos(12\tau)\right] \\
&= \lim_{\tau\to\infty} 18 + \frac{2}{6 + (t + \tau)^2}\left[1 + 4\cos(12\tau)\right] \\
&= \lim_{\tau\to\infty} 18 + \frac{\frac{2}{\tau^2}}{\frac{6}{\tau^2} + (\frac{t}{\tau} + 1)^2}\left[1 + 4\cos(12\tau)\right] \\
&= 18 + \frac{0}{0 + (0 + 1)^2}\left[1 + 4\cos 12\tau\right] \\
&= 18 + 0 = 18
\end{aligned}
$$

$\therefore \overline{X} = \pm\sqrt{18}$

b) No.

c)

$$P_{XX} = E[X^2(t)] = R_{XX}(0)$$
$$= \lim_{\tau\to\infty} 18 + \frac{2}{6+t^2}\left[1 + 4\cos(12\tau)\right]$$
$$= 18 + \frac{2}{6+t^2}\left[1 + \cos 0\right]$$
$$= 18 + \frac{10}{6+t^2} \quad \because \text{the total power at } t = 0 \text{ and } \tau = 0 \text{ So,}$$
$$= 18 + \frac{10}{6} = \frac{118}{6} = \frac{59}{3}\,Watts.$$

Problem 4: Assume that $X(t)$ is a WSS random process with an auto correlation function $R_{XX}(\tau) = e^{-\alpha|\tau|}$. Determine the second moment of the random variable $X(8) - X(5)$.

Solution: We know that $E[X(t)X(t+\tau)] = R_{XX}(\tau); \quad R_{XX}(0) = E[X^2(t)]$

The second central moment of the r.v X is given by $E[X^2(t)]$.

$$E\left[(X(8) - X(5))^2\right] = E[X^2(8)] + E[X^2(5)] - 2E[X^2(8)X^2(5)]$$
$$= E[X^2(8)] + E[X^2(5)] - 2E[X^2(5)X^2(5+3)]$$
$$= 1 + 1 - 2R_{XX}(0) \qquad \because E[X^2(t)] = R_{XX}(0) = e^0 = 1$$
$$= 2(1 - e^{-3\alpha})$$

Problem 5: The autocorrelation of a independent random process is given by $R_{XX}(\tau) = e^{-\alpha|\tau|}$. Find the auto correlation of a random process $Y(t) = X(t)\cos(\omega t + \vartheta)$, where ϑ is a random variable which is uniformly distributed with in 0 to 2π.

Solution: Given that $X(t)$ is independent

$$\text{Given } R_{XX}(\tau) = \begin{cases} e^{-\alpha\tau}; & \tau < 0 \\ e^{\alpha\tau}; & \tau > 0 \end{cases}$$

The auto correlation of a function $Y(t)$ is

$$R_{YY}(\tau) = E\left[Y(t)Y(t+\tau)\right]$$
$$= E\left[X(t)\cos(\omega t + \vartheta)X(t+\tau)\cos(\omega(t+\tau) + \vartheta)\right]$$
$$= E\left[X(t)X(t+\tau).\frac{1}{2}.2\cos(\omega t + \vartheta)\cos(\omega t + \omega\tau) + \vartheta)\right]$$
$$= E\left[X(t)X(t+\tau)\right].\frac{1}{2}.E\left[2\cos(\omega t + \vartheta)\cos(\omega t + \omega\tau) + \vartheta)\right]$$
$$= \frac{R_{XX}(\tau)}{2}E\left[\cos(2\omega t + 2\vartheta + \omega\tau) + \cos(\omega\tau)\right]$$
$$R_{YY}(\tau) = \frac{R_{XX}(\tau)}{2}\left[E[\cos(2\omega t + 2\vartheta + \omega\tau)] + E[\cos\omega\tau]\right]$$

$$E[\cos(2\omega t + 2\vartheta + \omega\tau)] = \int_{\vartheta=0}^{2\pi} \cos(2\omega t + 2\vartheta + \omega\tau)\,d\vartheta$$

$$= \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\vartheta + \omega\tau)}{2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(2\omega t + \omega\tau)}{2} - \frac{\sin(2\omega t + \omega\tau)}{2}\right]$$

$$= 0$$

$$E[\cos\omega\tau] = \frac{1}{2\pi}\int_{\vartheta=0}^{2\pi} \cos\omega\tau.1.d\vartheta$$

$$= \frac{1}{2\pi}\cos\omega\tau \int_{\vartheta=0}^{2\pi} 1.d\vartheta$$

$$= \frac{1}{2\pi}\cos\omega\tau \left[\vartheta\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\cos\omega\tau[2\pi - 0] = \frac{1}{2\pi}\cos\omega\tau[2\pi]$$

$$= \cos\omega\tau$$

The above two values of $E[\cos(2\omega t+2\vartheta+\omega\tau)]$ and $E[\cos\omega\tau]$; substitute in $R_{YY}(\tau)$.

$$\therefore R_{YY}(\tau) = \frac{R_{XX}(\tau)}{2}\left[0 + \cos\omega\tau\right]$$

$$= \frac{R_{XX}(\tau)}{2}\cos\omega\tau$$

$$\therefore R_{YY}(\tau) = \frac{e^{-a|\tau|}}{2}\cos\omega\tau$$

Problem 6: Let $X(t) = A\cos(\omega t + \vartheta)$, where the pdf of ϑ is

$$\text{Given } f_\vartheta(\vartheta) = \begin{cases} \frac{1}{2\pi}; & -\pi \le \vartheta \le \pi \\ 0; & Else\ \ where \end{cases}$$

Show that $X(t)$ is a stationary random process and find total power?

7.4 Cross Correlation

The correlation between two random variables which are obtain from two different random process is called cross correlation.

Let two random process $X(t)$ and $Y(t)$ with random variable $X(t_1)$ and $Y(t_2)$. The cross correlation can be defined as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$
$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

7.4.1 Properties:

1. If $X(t)$ and $Y(t)$ are orthogonal process then $R_{XY}(\tau) = 0$

2. If $X(t)$ and $Y(t)$ are independent and WSS random process, then,

$$R_{YY}(\tau) = E[X]E[Y] = \overline{X}\,\overline{Y}$$

3. If two random process $X(t)$ and $Y(t)$ have zero mean and independent, then

$$\lim_{\tau\to\infty} R_{XY}(\tau) = 0$$

4. The cross correlation function is even function i.e., $R_{XY}(\tau) = R_{XY}(-\tau)$

Proof.

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

Let $\tau = -\tau$

$$R_{XY}(-\tau) = E[X(t)Y(t-\tau)]$$

Let $t - \tau = u \longrightarrow t = u + \tau$

$$R_{XY}(-\tau) = E[X(u+\tau)Y(u)]$$
$$= R_{XY}(\tau)$$

□

5. The maximum value is obtained at origin $|R_{XY}| = \sqrt{R_{XX}(0)R_{YY}(0)}$

Proof. Let $\left[\frac{X(t_1)}{\sqrt{R_{XX}(0)}} \pm \frac{Y(t_2)}{\sqrt{R_{YY}(0)}}\right]^2 \geq 0$

$$E\left[\frac{X(t)}{\sqrt{R_{XX}(0)}} \pm \frac{Y(t+\tau)}{\sqrt{R_{YY}(0)}}\right]^2 \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)} + \frac{Y^2(t+\tau)}{R_{YY}(0)} + 2\frac{X(t)Y(t+\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}\right] \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)}\right] + E\left[\frac{Y^2(t+\tau)}{R_{YY}(0)}\right] + 2E\left[\frac{X(t)Y(t+\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}\right] \geq 0$$

$$\left[\frac{R_{XX}(0)}{R_{XX}(0)}\right] + \left[\frac{R_{YY}(0)}{R_{YY}(0)}\right] + 2\left[\frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}\right] \geq 0$$

$$\left[\frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}\right] \leq 1$$

$$\therefore R_{XY}(\tau) \leq \sqrt{R_{XX}(0)R_{YY}(0)}$$

□

7.5 Covariance Function

We know that co-variance of two random variables X and Y are

$$C_{XY} = \mu_{11} = E[(x - \overline{X})(y - \overline{Y})] = R_{XY} - \overline{X}\,\overline{Y} = E[XY] - \overline{X}\,\overline{Y}$$

7.5.1 Auto covariance

Let $X(t)$ be a random process with random variable obtained at t_1 and t_2, the covariance can be defined as

$$C_{X_1X_2}(\tau) = E\left[\left(X(t_1) - \overline{X(t_1)}\right)\left(X(t_2) - \overline{X(t_2)}\right)\right] = C_{X_1X_2}(t_1, t_2)$$

$$= E\left[\left(X(t) - \overline{X(t)}\right)\left(X(t+\tau) - \overline{X(t+\tau)}\right)\right]$$

$$= E\left[X(t)X(t+\tau) - X(t)\overline{X(t+\tau)} - \overline{X(t)}X(t+\tau) + \overline{X(t)}\,\overline{X(t+\tau)}\right]$$

$$= E[X(t)X(t+\tau)] - \overline{X(t+\tau)}E[X(t)] - \overline{X(t)}E[X(t+\tau)] + \overline{X(t)}\,\overline{X(t+\tau)}$$

$$= E[X(t)X(t+\tau)] - \overline{X(t)}\,\overline{X(t+\tau)} - \overline{X(t)}\,\overline{X(t+\tau)} + \overline{X(t)}\,\overline{X(t+\tau)}$$

$$C_{X_1X_2}(\tau) = R_{XX}(\tau) - \overline{X(t)}\,\overline{X(t+\tau)}$$

If $X(t)$ is WSS then $C_{X_1X_2}(\tau) = R_{XX}(\tau) - \overline{X}^2(t)$

In general $C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2(t)$

7.5.2 Properties:

1. If $X(t_1)$ and $X(t_2)$ are orthogonal, then $E[X(t_1)X(t_2)] = 0 \longrightarrow R_{XX}(0) = 0$
   $\therefore C_{XX}(\tau) = -\overline{X}^2(t)$

2. If $X(t_1)$ and $X(t_2)$ are independent then, $C_{XX}(\tau) = 0$

3. If $\tau = 0$ and $C_{XX}(\tau) = 0$ then

$$
\begin{aligned}
C_{XX}(0) &= R_{XX}(0) - \overline{X}^2(t) \\
&= E[X^2(t)] - \overline{X}^2(t) \\
C_{XX}(0) &= \sigma^2_X
\end{aligned}
$$

7.5.3 Cross covariance

Let $X(t)$ and $Y(t)$ are two random process. The covariance between two random variables $X(t_1)$ and $X(t_2)$ which are obtained from $X(t)$ and $Y(t)$ random process. The cross covariance can be written as

$$
\begin{aligned}
C_{XY}(\tau) &= E[X(t)Y(t+\tau)] - \overline{X}\,\overline{Y} \\
C_{XY}(\tau) &= R_{XX}(\tau) - \overline{X}\,\overline{Y} \\
C_{XY}(i) &= R_{XX}(i) - \overline{X}^2
\end{aligned}
$$

7.6 The Time Averages of Random Process

Let $X(t)$ be the random process, the averages can be written as

- Mean time:

$$
\begin{aligned}
A[X(t)] &= \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} X(t)dt \qquad \textbf{(or)} \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} X(t)dt
\end{aligned}
$$

- Time correlation

$$
\begin{aligned}
R_{XX}(\tau) &= A[X(t)X(t+\tau)] \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} X(t)X(t+\tau)dt
\end{aligned}
$$

7.6.1 The statistical averages of random process

1. Statistical mean: $E[X(t) = \int X(t).PDF$

2. Statistical correlation: $R_{XX}(\tau) = E[X(t)X(t+\tau)] = \int X(t)X(t+\tau).PDF$

3. Ergodic random process: If statistical averages is equal to time averages then it is called Ergodic r.p.

4. Mean Ergodic random process: If only statistical mean is equal to time mean then it is called Mean Ergodic r.p.

5. Correlation Ergodic random process: If only statistical correlation is equal to time correlation then it is called Correlation Ergodic r.p.

Problem 1: A random process $X(t) = A\cos(\omega_0 t + \vartheta)$ is stationary if A and ω_0 are constants and ϑ is a uniformly distributed variable on the interval $(0, 2\pi)$. Prove that $X(t)$ is Ergodic random process.

Solution: Given $X(t) = A\cos(\omega_0 t + \vartheta)$; where A and ω_0 are constants and $\vartheta \longrightarrow (0, 2\pi)$; $f_\vartheta(\vartheta) = \frac{1}{2\pi}$. The ϑ is uniformly distributed between 0 to 2π. The distribution shown in Fig. 7.12.

Fig. 7.12

If mean value and auto correlation function of r.p is a function of time, 't' then it is not stationary.

1. Expectation or mean value:

$$E[X(t)] = \overline{X(t)} = \int_{\vartheta=0}^{2\pi} x(t) f_\vartheta(\vartheta)\, d\vartheta$$

$$= \int_{\vartheta=0}^{2\pi} A\cos(\omega_0 t + \vartheta).\frac{1}{2\pi} d\vartheta$$

$$= \frac{A}{2\pi}\int_{\vartheta=0}^{2\pi} \cos(\omega_0 t + \vartheta)\, d\vartheta$$

$$= \frac{A}{2\pi}\left[\sin(\omega_0 t + \vartheta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}[\sin(\omega_0 t + 2\pi) - \sin\omega_0 t]$$

$$= \frac{A}{2\pi}[\sin\omega_0 t - \sin\omega_0 t]$$

$$= 0$$

$\therefore\ E[X(t)] = \overline{X(t)} = 0$. It is constant.

2. Correlation:

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$= \int_{\vartheta=0}^{2\pi} x(t)X(t+\tau)f_{\vartheta}(\vartheta)d\vartheta$$
$$= \int_{\vartheta=0}^{2\pi} A\cos(\omega_0 t+\vartheta).A\cos(\omega_0(t+\tau)+\vartheta)\frac{1}{2\pi}d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} 2\cos(\omega_0 t+\vartheta).\cos(\omega_0 t+\omega_0\tau+\vartheta)d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} \cos(\omega_0 t+\vartheta+\omega_0 t+\omega_0\tau+\vartheta)+\cos(\omega_0 t+\vartheta-\omega_0 t-\omega_0\tau-\vartheta)d\vartheta$$
$$= \frac{A^2}{4\pi}\int_{\vartheta=0}^{2\pi} \cos(2\omega_0 t+2\vartheta+\omega_0\tau)+\cos(\omega_0\tau)d\vartheta$$
$$= \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t+2\vartheta+\omega_0\tau)}{2}\right]_0^{2\pi} + \frac{A^2}{4\pi}[\cos(\omega_0\tau)]\left[\vartheta\right]_0^{2\pi}$$
$$= \frac{A^2}{4\pi}[\cos(\omega_0\tau)]\left[\vartheta\right]_0^{2\pi} + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t+4\pi+\omega_0\tau)-\sin(2\omega_0 t+\omega_0\tau)}{2}\right]$$
$$= \frac{A^2}{4\pi}\cos(\omega_0\tau)[2\pi-0] + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t+\omega_0\tau)-\sin(2\omega_0 t+\omega_0\tau)}{2}\right]$$
$$= \frac{A^2}{2}\cos\omega_0\tau + 0$$
$$= \frac{A^2}{2}\cos\omega_0\tau$$

$\therefore R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$. This solution does not contain variable 't'. So, both $E[X(t)]$ and $E[X(t)X(t+\tau)]$ are constant, then it is WSS.

CHAPTER 8

Spectral Characteristics

8.1 Spectral Representation

In previous sections studied the characteristics of random process in time domain. The characteristics of random process can be represented in frequency domain also and the function obtained in frequency domain is called the spectrum of random signal and measured in '*volts/Hertz*'.

Let $X(t)$ be a random process as shown in Fig.

The random process $X(t)$ and $X_T(t)$ be defined as that portion of $X(t)$ between $-T$ to $+T$ i.e.,

$$X_T(t) = \begin{cases} X(t); & -T < t < t \\ 0; & \text{otherwise} \end{cases}$$

Fourier transforms are very useful in spectral in spectral representation of the random signals. For example, consider a random signal $x(t)$, the Fourier transform of $x(t)$ is $X(\omega)$ is given by

$$X(\omega) = F\ x(t) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

This function $X(\omega)$ is considered to the voltage density specturam of $x(t)$.; But, the problem is that $X(\omega)$ may not exist for many functions of a random process. Therefore, the spectral representation of random process utilizing a voltage density spectrum is not feasible always.

In such situation, we go for the power density spectrum of a random process which is defined as the function which results when the power in the random process is described as a function of frequency.

8.2 Power Spectral Density (PSD)

Let $X(t)$ be a random process (r.p) as shown in Fig.

The random process $X(t)$ between $-T$ to $+T$ can be written as

$$X_T(t) = \begin{cases} X(t); & -T \le t \le T \\ 0; & \text{elsewhere} \end{cases}$$

The Fourier Transform of $X_T(t)$ can be written as

$$F[X(t)] = X_T(\omega) = \int_{t=-\infty}^{\infty} X_T(t) e^{-j\omega t}\, dt$$

$$= \int_{t=-T}^{T} X_T(t) e^{-j\omega t}\, dt$$

$$\therefore X_T(\omega) = \int_{t=-T}^{T} X(t) e^{-j\omega t}\, dt$$

The energy and power of a random process:

1. Time domain

- Energy $E = \int_{t=-T}^{T} X^2(t)\, dt = \int_{t=-T}^{T} X_T^2(t)\, dt$
- Power $P = \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} X^2(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} X_T^2(t)\, dt$

2. Frequency domain

- Energy $E = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X^2(\omega)|\, d\omega = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X_T^2(\omega)|\, d\omega$
- Power $P = \lim_{T\to\infty} \frac{1}{2T} \times \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X^2(\omega)|\, d\omega = \lim_{T\to\infty} \frac{1}{2T} \times \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X_T(\omega)|^2\, d\omega$

The average power of random process can be written as

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} E\left[X^2(t)\right] dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \times \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} E\left[X_T^2(\omega)\right] d\omega$$

$$= \frac{1}{2\pi} \lim_{T\to\infty} \int_{\omega=-\infty}^{\infty} \frac{E\left[X_T^2(\omega)\right]}{2T}\, d\omega$$

$$= A\left[E\left[X^2(t)\right]\right] dt \qquad \because \lim_{T\to\infty} \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \frac{E\left[X_T^2(\omega)\right]}{2T}\, d\omega = A\left[E\left[X^2(t)\right]\right]$$

$$\therefore P_{XX} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

where $S_{XX}(\omega)$ is called Power Spectral Density (PSD) or Power Density Spectrum (PDS) and given by

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

8.2.1 Wiener Kinchin Relation

The Wiener Kinchin relation says that Power Spectral Density (PSD) and Auto-correlation function from the Fourier Transform pair.

$$S_{XX}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega\tau}\, d\omega$$

$$\boxed{\therefore R_{XX}(\tau) \overset{F}{\leftrightarrow} S_{XX}(\omega)}$$

Proof. Let $X(t)$ be the random process with PSD of

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega)X_T(\omega)\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left\{\int_{t_1=-\infty}^{\infty} X(t_1)e^{j\omega t_1}\, dt_1 \cdot \int_{t_2=-\infty}^{\infty} X(t_2)e^{-j\omega t_1}\, dt_2\right\}$$

Where $X(t_1)$ and $X(t_2)$ are two random variables obtained from random process $X(t)$ as $t = t_1$ and $t = t_2$

$$X_T^*(\omega) = F[X(t_1)]$$

$$X_T(\omega) = F[X(t_2)]$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{t_1=-T}^{T}\int_{t_2=-T}^{T} E\left[X(t_1)X(t_2)\right] e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

let $t_1 = T, t_2 = t_1 + \tau \Rightarrow \tau = t_2 + t_1$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} \int_{\tau=T-t}^{T+t} E\left[X(t)X(t+\tau)\right] e^{-j\omega\tau}\, dt\, d\tau$$

$$= \int_{\tau=-T}^{T} \left[\lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} E\left[X(t)X(t+\tau)\right] dt\right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=-T}^{T} A\left[R_{XX}(\tau)\right] e^{-j\omega\tau}\, d\tau$$

When the random process $X(t)$ is atleast Wide Sense Stationary random process (WSS rp), we can write

$$A\left[R_{XX}(\tau)\right] = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\boxed{S_{XX}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau = F\left[R_{XX}(\tau)\right]}$$

$$\boxed{R_{XX}(\tau) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega = F^{-1}\left[S_{XX}(\omega)\right]}$$

□

8.2.2 Properties of Power Spectral Density (PSD)

1. Power spectral density is non-negative function.

   Proof.

   $$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\left|X_T(\omega)\right|^2\right]$$

   Using the above equation we can say that PSD is a non-negative function. □

2. Power spectral density is a real valued function.

   Proof.

   $$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\left|X_T(\omega)\right|^2\right]$$

   Using the above equation we can say that PSD is a real valued function. □

3. Power spectal density is even function

Proof.

$$S_{XX}(-\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau \tag{8.1}$$

Let $\omega = -\omega$

$$S_{XX}(-\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)e^{+j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \underbrace{R_{XX}(\tau)}_{\text{even}} e^{-j\omega(-\tau)}\, d\tau$$

Auto-correlation function is even function $R_{XX}(\tau) = R_{XX}(-\tau)$

$$S_{XX}(-\omega) = \int_{\tau=-\infty}^{\infty} \underbrace{\overline{R_{XX}(-\tau)}}_{\text{even}} e^{-j\omega\tau}\, d\tau \tag{8.2}$$

From equation (8.1) and (8.2)

$$\boxed{S_{XX}(-\omega) = S_{XX}(\omega)}$$

□

4. The total area of the auto-correlation function is equal to DC component (or) average value of a random process.

Proof. From Wiener Kinchin relation

$$S_{XX}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

The DC or average value of random process can be obtained by subtituting $\omega = 0$ in PSD

$$S_{XX}(0) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

□

5. The total power or mean square value of random process is equal to the total area of PSD.

Proof. From Wiener Kinchin relation

$$R_{XX}(\tau) = frac12\pi \int_{\omega=-\infty}^{\infty} S_{XX}(\tau)e^{j\omega\tau}\ d\omega$$

The total power of random process can be obtained by substituting $\tau = 0$

$$R_{XX}(\tau) = E\ X(t)X(t+\tau)$$

If $\tau = 0$ then $R_{XX}(0) = E\ X^2(t)$

$$\therefore\ R_{XX}(\tau) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\ d\omega$$

□

NOTE:

- Total power

$$\therefore\ R_{XX}(0) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(f)\ df = E[X^2(t)];$$

- DC power or Average power (power at zero freq)

$$S_{XX}(0) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau)\ d\tau$$

6. The PSD of derivation of the random process $\frac{d}{dt}X(t)$ is ω^2 times the PSD of the random process.

S$_{\dot{X}\dot{X}}(\omega) = \omega^2$ S$_{XX}(\omega)$; . $\longrightarrow$ denotes derivative

Proof.

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{1}{2T}E\left[\left|X_T(\omega)\right|^2\right]$$

$$X_T(\omega) = \int_{t=-T}^{T} X_T(t)e^{-j\omega t}\ dt$$

$$\frac{d}{dt}X_T(\omega) = \dot{X}_T(\omega) = \int_{t=-T}^{T} X_T(t)e^{-j\omega t}\ (-j\omega)\ dt$$

$$\dot{X}_T(\omega) = -j\omega \cdot X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T\to\infty} \frac{E\left[\left|\dot{X}_T(\omega)\right|^2\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{E\left[\left|-j\omega X_T(\omega) \cdot -j\omega X_T(\omega)\right|\right]}{2T}$$

$$= \omega^2 \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \omega^2 S_{XX}(\omega)$$

$$\boxed{S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)}$$

□

Problem 1: Find the PSD of Auto-correlation function $R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$ and plot both Auto-correlation (ACF) and PSD.

Solution:

$$R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

$$= \frac{A^2}{2}\left[\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right]$$

$$= \frac{A^2}{4}e^{j\omega_0\tau} + \frac{A^2}{4}e^{-j\omega_0\tau}$$

The PSD is a Fourier transform of Autocorrelation function.

$$S_{XX}(\omega) = F\,[R_{XX}(\tau)]$$

$$= F\left[\frac{A^2}{4}e^{j\omega_0\tau} + \frac{A^2}{4}e^{-j\omega_0\tau}\right]$$

$$= \frac{A^2}{4}F\left[e^{j\omega_0\tau}\right] + \frac{A^2}{4}F\left[e^{-j\omega_0\tau}\right]$$

$$= \frac{A^2}{4} \times 2\pi\delta(\omega - \omega_0) + \frac{A^2}{4} \times 2\pi\delta(\omega + \omega_0)$$

$$= \frac{A^2}{2}\pi\,[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\boxed{\therefore S_{XX}(\omega) = \frac{A^2}{2}\pi\,[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}$$

$$\delta(\omega) = \begin{cases} 1; & \omega = 0 \\ 0; & \omega \neq 0 \end{cases}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \delta(\omega)e^{j\omega\tau}\, d\omega$$

$$F\left\{F^{-1}[\delta(\omega)]\right\} = F\left[\frac{1}{2\pi}(1)\right]$$

$$\delta(\omega) = \frac{1}{2\pi}F[1]$$

$$2\pi\delta(\omega) = F[1]$$

If $\omega = \omega_0$

$$2\pi\delta(\omega - \omega_0) = F\left[e^{j\omega_0\tau}\right]$$

If $\omega = -\omega_0$

$$2\pi\delta(\omega + \omega_0) = F\left[e^{-j\omega_0\tau}\right]$$

Let $X(\omega) = \delta(\omega - \omega_0)$

$$x(t) = F^{-1}\left[X(\omega)\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\, d\omega$$

From sampling property of the impulse function

$$F^{-1}\left[\delta(\omega - \omega_0)\right] = \frac{1}{2\pi}\left[e^{j\omega t}\right]_{\omega=\omega_0}$$

$$= \frac{1}{2\pi}e^{j\omega_0 t}$$

$$= \frac{1}{2\pi}F\left[e^{j\omega_0 t}\right]$$

$$\boxed{F\left[e^{j\omega_0 t}\right] = 2\pi\delta(\omega - \omega_0)}$$

Problem: 2 Find the PSD of Autocorrelation function

$$R_{XX}(\tau) = \begin{cases} A\left[1 + \frac{|\tau|}{T}\right]; & -T \leq \tau \leq T \\ 0; & \text{otherwise} \end{cases}$$

Solution: Given

$$R_{XX}(\tau) = \begin{cases} A\left[1 + \frac{\tau}{T}\right]; & -T \leq \tau \leq 0 \\ A\left[1 - \frac{\tau}{T}\right]; & 0 \leq \tau \leq T \\ 0; & \text{otherwise} \end{cases}$$

Method-1:

Using ramp function $r(\tau)$

$$R_{XX}(\tau) = \frac{A}{T}r(\tau + T) - \frac{2A}{T}r(\tau) + \frac{A}{T}r(\tau - T)$$

First derivative of $R_{XX}(\tau)$ with respect to 'τ'

$$\frac{d}{d\tau}R_{XX}(\tau) = \frac{A}{T}U(\tau + T) - \frac{2A}{T}U(\tau) + \frac{A}{T}U(\tau - T)$$

Second derivative of $R_{XX}(\tau)$ with respect to 'τ'

$$\frac{d^2}{d\tau^2}R_{XX}(\tau) = \frac{A}{T}\delta(\tau + T) - \frac{2A}{T}\delta(\tau) + \frac{A}{T}\delta(\tau - T) \quad (8.3)$$

We know that $\frac{d^2}{d\tau} R_{XX}(\tau^2) = (j\omega)^2 \; S_{XX}(\omega)$

Now, using differentiation and shifting property,

If $x(t) \leftrightarrow X(\omega)$ then $\frac{d}{dt}x(t) \leftrightarrow j\omega \; X(\omega)$;

$\frac{d^2}{dt^2}x(t) \leftrightarrow (j\omega)^2 \; X(\omega)$ and

$x(t - t_0) \leftrightarrow X(\omega)\; e^{-j\omega t_0}$

From equation (8.3), apply Fourier transform on both sides

$$(j\omega)^2 S_{XX}(\omega) = \frac{A}{T} F\; \delta(\tau + T) \; - \frac{2A}{T} F\; \delta(\tau) \; + \frac{A}{T} F\; \delta(\tau - T)$$

$$-\omega^2 S_{XX}(\omega) = \frac{A}{T}\, e^{j\omega T} - \frac{2A}{T} + \frac{A}{T}\, e^{-j\omega T}$$

$$= \frac{2A}{T}\left(\frac{e^{j\omega T} + e^{-j\omega T}}{2} - 1\right)$$

$$= \frac{2A}{T}\,[\cos\omega T - 1]$$

$$S_{XX}(\omega) = \frac{2A}{T\omega^2}[1 - \cos\omega T]$$

$$= \frac{4A}{\omega^2 T}\sin^2\left(\frac{\omega T}{2}\right) \qquad \because \sin^2\vartheta = \frac{1 - \cos 2\vartheta}{2}$$

$$= AT\left(\frac{\sin^2\frac{\omega T}{2}}{\frac{\omega^2 T^2}{2^2}}\right)$$

$$= AT\left[\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right]^2$$

$$= AT\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$$

$$\boxed{S_{XX}(\omega) = AT\;\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)}$$

Method-2:

$$S_{XX}(\omega) = F\;R_{XX}(\tau) \; = \int_{\tau=-T}^{T} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau$$

$$= \int_{\tau=-T}^{0} A\left(1 + \frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau + \int_{\tau=0}^{T} A\left(1 - \frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau$$

$$= \int_{\tau=0}^{T} A\left(1 - \frac{\tau}{T}\right)e^{+j\omega\tau}\,d\tau + \int_{\tau=0}^{T} A\left(1 - \frac{\tau}{T}\right)e^{-j\omega\tau}\,d\tau$$

$$= \int_{\tau=0}^{T} A\left(1 - \frac{\tau}{T}\right)\left[e^{j\omega\tau} + e^{-j\omega\tau}\right]d\tau$$

$$= 2A\int_{\tau=0}^{T}\left(1-\frac{\tau}{T}\right)\left[\frac{e^{j\omega\tau}+e^{-j\omega\tau}}{2}\right]d\tau$$

$$= 2A\int_{\tau=0}^{T}\left(1-\frac{\tau}{T}\right)\cos\omega\tau\, d\tau$$

$$= 2A\int_{\tau=0}^{T}\cos\omega\tau\, d\tau - 2A\int_{\tau=0}^{T}\frac{\tau}{T}\cos\omega\tau\, d\tau$$

$$= 2A\left[\frac{\sin\omega\tau}{\omega}\right]_{\tau=0}^{T} - \frac{2A}{T}\left[\tau\frac{\sin\omega\tau}{\omega} - \int 1\cdot\frac{\sin\omega\tau}{\omega}\right]_{\tau=0}^{T}$$

$$= 2A\left[\frac{\sin\omega\tau}{\omega} - 0\right] - \frac{2A}{T}\left[\tau\frac{\sin\omega\tau}{\omega} + \tau\frac{\cos\omega\tau}{\omega^2}\right]_{\tau=0}^{T}$$

$$= 2A\frac{\sin\omega T}{\omega} - \frac{2A}{T}\left[T\frac{\sin\omega T}{\omega} + \frac{\cos\omega T}{\omega^2} - \left(0+\frac{1}{\omega^2}\right)\right]$$

$$= 2A\frac{\sin\omega T}{\omega} - 2A\frac{\sin\omega T}{\omega} - \frac{2A}{T}\frac{\cos\omega T}{\omega^2} + \frac{2A}{T}\cdot\frac{1}{\omega^2}$$

$$= \frac{2A}{T\omega^2}[1-\cos\omega T]$$

$$= \frac{4A}{\omega^2 T}\sin^2\frac{\omega T}{2} \qquad \because \sin^2\vartheta = \frac{1-\cos 2\vartheta}{2}$$

$$= AT\frac{\sin^2\frac{\omega T}{2}}{\frac{\omega^2T^2}{4}}$$

$$= AT\left[\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right]^2$$

$$= AT\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$$

$$\therefore \boxed{S_{XX}(\omega) = AT\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right)}$$

Problem: 3 The autocorrelation function of the random telegraph process is given by $R_{XX}(\tau) = e^{-2\alpha|\tau|}$. Find the power spectral density (PSD) ? Solution: Given $R_{XX}(\tau) = e^{-2\alpha|\tau|}$

$$R_{XX}(\tau) = \begin{cases} e^{2\alpha\tau}; & -\infty \le \tau \le 0 \\ e^{-2\alpha\tau}; & 0 \le \tau \le \infty \end{cases}$$

$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

$$S_{XX}(f) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau) e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{0} e^{2\alpha\tau}\, e^{-j2\pi f\tau}\, d\tau + \int_{\tau=0}^{\infty} e^{-2\alpha\tau}\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{0} e^{(2\alpha - j2\pi f)\tau}\, d\tau + \int_{\tau=0}^{\infty} e^{(-2\alpha - j2\pi f)\tau}\, d\tau$$

$$= \left[\frac{e^{(2\alpha - j2\pi f)\tau}}{2\alpha - j2\pi f}\right]_{-\infty}^{0} + \left[\frac{e^{(-2\alpha - j2\pi f)\tau}}{-2\alpha - j2\pi f}\right]_{0}^{\infty}$$

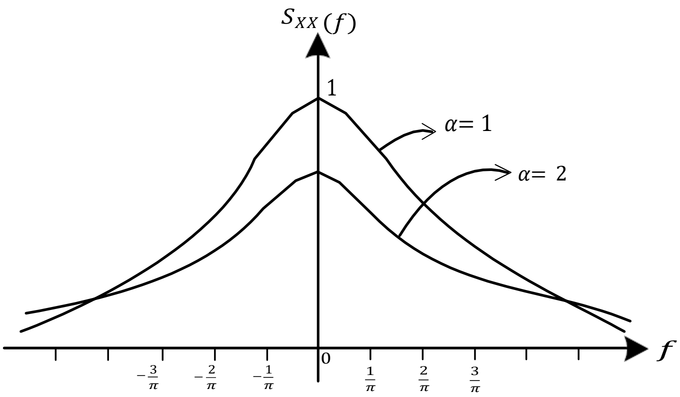
$$= \frac{1}{2\alpha - j2\pi f}[1 - 0] + \frac{1}{-2\alpha - j2\pi f}[0 - 1]$$

$$= \frac{1}{2\alpha - j2\pi f} + \frac{1}{2\alpha + j2\pi f}$$

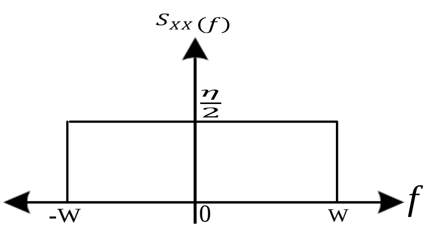
$$= \frac{2\alpha + j2\pi f + 2\alpha - j2\pi f}{(2\alpha)^2 - (j2\pi f)^2}$$

$$= \frac{4\alpha}{4\alpha^2 - 4\pi^2 f^2}$$

$$\therefore S_{XX}(f) = \frac{4\alpha}{4\alpha^2 - 4\pi^2 f^2}$$



Problem: 4 The power spectral density of a WSS white noise whose frequency components are limited to $-W \leq f \leq W$ is shown in the Fig.



(a) Find average power of $X(t)$?

(b) Find auto-correlation of $X(t)$?

Solution:

(a) Average power

$$E\left[X^2(t)\right] = \int_{f=-W}^{W} S_{XX}(f)\, df$$

$$= \int_{f=-W}^{W} \frac{\eta}{2} df$$

$$= \frac{\eta}{2}[f]_{-W}^{W} = \frac{\eta}{2}[2W]$$

$$= \eta W$$

$$\therefore E\left[X^2(t)\right] = \eta W$$

(b) Auto correlationfor this process is

$$R_{XX}(\tau) = \int_{f=-\infty}^{\infty} S_{XX}(f) e^{j2\pi f\tau}\, df$$

$$= \int_{f=-W}^{W} \frac{\eta}{2} e^{j2\pi f\tau}\, df$$

$$= \frac{\eta}{2}\left[\frac{e^{j2\pi f\tau}}{j2\pi\tau}\right]_{-W}^{W}$$

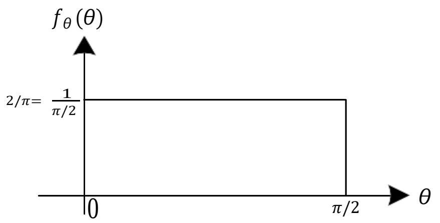
$$= \frac{\eta}{2}\left[\frac{e^{j2\pi W\tau} - e^{j2\pi W\tau}}{j2\pi W\tau}\right]$$

$$= \frac{\eta}{2\pi\tau}\sin(2\pi W\tau)$$

$$= \eta W \cdot \frac{\sin(2\pi W\tau)}{2\pi W\tau}$$

$$R_{XX}(\tau) = \eta W \operatorname{sinc}(2W\tau) \qquad \because \frac{\sin \pi x}{\pi x} = \operatorname{sinc} x$$

Problem: 5 For the random process $X(t) = A\sin(\omega_0 t + \vartheta)$, where A and ω_0 are real constants and ϑ is a random variable distributed uniformly in the interval $0 < \vartheta < \frac{\pi}{2}$. Find the average power P_{XX} in $X(t)$?



Solution:

First Approach:

$$E\left[X^2(t)\right] = E\left[A^2\sin^2(\omega_0 t + \vartheta)\right] \qquad \because \sin^2\vartheta = \frac{1 = \cos 2\vartheta}{2}$$

$$= E\left[\frac{A^2}{2} - \frac{A^2}{2}\cos(2\omega_0 t + 2\vartheta)\right]$$

$$= \frac{A^2}{2} - \frac{A^2}{2}\int_{\vartheta=0}^{\frac{\pi}{2}} \cos(2\omega_0 t + 2\vartheta)\cdot\frac{2}{\pi}\, d\vartheta$$

$$= \frac{A^2}{2} - \frac{A^2}{\pi}\left[\frac{\sin(2\omega_0 t + 2\vartheta)}{2}\right]_{\vartheta=0}^{\frac{\pi}{2}}$$

$$= \frac{A^2}{2} - \frac{A^2}{2\pi}[\sin(2\omega_0 t + \pi) - \sin 2\omega_0 t]$$

$$= \frac{A^2}{2} - \frac{A^2}{2\pi}\{[\sin(2\omega_0 t)\cos\pi - \cos(2\omega_0 t)\sin\pi] - \sin 2\omega_0 t\}$$

$$= \frac{A^2}{2} - \frac{A^2}{2\pi}[-\sin 2\omega_0 t - \sin 2\omega_0 t]$$

$$= \frac{A^2}{2} + \frac{A^2}{\pi}\sin 2\omega_0 t$$

Since $E[X^2(t)]$ is time dependent. So, $X(t)$ is not WSS random process. Finally we perform time averages is

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{t=-T}^{T} E[X^2(t)]dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{t=-T}^{T} \left(\frac{A^2}{2} + \frac{A^2}{\pi}\sin 2\omega_0 t\right) dt$$

$$= \frac{A^2}{2}$$

$$\boxed{\therefore P_{XX} = \frac{A^2}{2}}$$

Second Approach:

$$X_T(\omega) = \int_{t=-\infty}^{\infty} X_T(t)e^{-j\omega t}\, dt$$

$$= \int_{t=-T}^{T} A\sin(\omega_0 t + \vartheta)\, e^{-j\omega t}\, dt$$

$$= A\int_{t=-T}^{T} \left(\frac{e^{j(\omega_0 t+\vartheta)} - e^{-j(\omega_0 t+\vartheta)}}{2j}\right) e^{-j\omega t}\, dt$$

$$= \frac{A}{2j}e^{j\vartheta}\int_{t=-T}^{T} e^{j(\omega_0-\omega)t}\, dt - \frac{A}{2j}e^{-j\vartheta}\int_{t=-T}^{T} e^{-j(\omega_0+\omega)t}\, dt$$

$$= \frac{A}{2j}e^{j\vartheta}\left[\frac{e^{j(\omega_0-\omega)t}}{j(\omega_0-\omega)}\right]_{t=-T}^{T} - \frac{A}{2j}e^{-j\vartheta}\left[\frac{e^{-j(\omega_0+\omega)t}}{-j(\omega_0+\omega)}\right]_{t=-T}^{T}$$

$$= \frac{A}{j(\omega_0-\omega)}e^{j\vartheta}\left(\frac{e^{j(\omega_0-\omega)T} - e^{-j(\omega_0-\omega)T}}{2j}\right) + \frac{A}{j(\omega_0+\omega)}e^{-j\vartheta}\left(\frac{e^{-j(\omega_0+\omega)T} - e^{j(\omega_0+\omega)T}}{2j}\right)$$

$$= \frac{ATe^{j\vartheta}}{j}\frac{\sin(\omega_0-\omega)T}{(\omega_0-\omega)T} - \frac{ATe^{-j\vartheta}}{j}\frac{\sin(\omega_0+\omega)T}{(\omega_0+\omega)T}$$

$$X_T(\omega) = jAT\left[e^{j\vartheta}\frac{\sin(\omega_0-\omega)T}{(\omega_0-\omega)T} + e^{-j\vartheta}\frac{\sin(\omega_0+\omega)T}{(\omega_0+\omega)T}\right]$$

$$|X_T(\omega)|^2 = jAT(-jAT)\left[e^{j\vartheta}\frac{\sin(\omega_0-\omega)T}{(\omega_0-\omega)T} + e^{-j\vartheta}\frac{\sin(\omega_0+\omega)T}{(\omega_0+\omega)T}\right]$$

$$\times \left[e^{-j\vartheta} \frac{\sin(\omega_0 - \omega)T}{(\omega_0 - \omega)T} + e^{j\vartheta} \frac{\sin(\omega_0 + \omega)T}{(\omega_0 + \omega)T} \right]$$

$$|X_T(\omega)|^2 = P_{XX} = \frac{A^2}{2}$$

$$\boxed{\therefore \text{ Average power } P_{XX} = \frac{A^2}{2}}$$

By comparing both two methods, the direct method (second method) is tedious. So, very easy to compute first method.

Problem: 6 For the stationary ergodic random process having the auto correlation function as shown in Fig,. Find (*a*) $E[X(t)]$ (*b*) $E[X^2(t)]$ (*c*) σ_X^2 of $R_{XX}(\tau)$

Solution:

(a) $(E[X(t)])^2 = \lim_{\tau\to\infty} R_{XX}(\tau) = 20$

$\therefore E[X(t)] = \sqrt{20}$

(b) $E[X^2(t)] = R_{XX}(0) = 50$

(c) $\sigma_X^2 = E[X^2(t)] - (E[X(t)])^2 = 50 - 20 = 30$

Problem: 7 Assume that an ergodic random process $X(t)$ has an auto-correlation function $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}[1 + 4\cos(12\tau)]$. Find $\overline{X(t)}$ and what is average power of $X(t)$?

Solution: (a) Square of Mean value:

$$\left[E[X(t)]\right]^2 = \lim_{\tau\to\infty} R_{XX}(\tau)$$

$$= \lim_{\tau\to\infty} 18 + \frac{2}{6+\tau^2}[1 + 4\cos(12\tau)]$$

$$\therefore (\overline{X(t)})^2 = 18$$

$$\Longrightarrow \boxed{\overline{X(t)} = \pm\sqrt{18}}$$

(b) Average power:

$$P_{XX} = E[X^2(t)] = R_{XX}(\tau = 0)$$

$$= 18 + \frac{2}{6+\tau^2}\left[1 + 4\cos(12\tau)\right]\Big|_{\tau=0}$$

$$= 18 + \frac{2}{6+0}\left[1 + 4(1)\right]$$

$$= 18 + \frac{10}{6} = \frac{118}{6}$$

$$\Longrightarrow \boxed{P_{XX} = \frac{59}{3} \text{ Watts}}$$

Problem: 8 Let $X(t) = A\cos(\omega_0 t + \vartheta)$, $f_\vartheta(\vartheta) = \begin{cases} \frac{1}{2\pi}; & -\pi \le \vartheta \le \pi \\ 0; & \text{elsewhere} \end{cases}$

and $Y(t) = B\cos(\omega_0 t)$; where $f_B(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}$; $-\infty \le b \le \infty$.

Find

$E[X(t)]$	$E[Y(t)]$	$C_{XX}(\tau)$
$E[X^2(t)]$	$E[Y^2(t)]$	$R_{XY}(\tau)$
σ_X^2	σ_Y^2	$C_{XY}(\tau)$
$R_{XX}(\tau)$	$R_{YY}(\tau)$	

Solution:

1.

$$
\begin{aligned}
E[X(t)] &= E[A\cos(\omega_0 t + \vartheta)] \\
&= AE[\cos\omega_0 t \sin\vartheta + \sin\omega_0 t\cos\vartheta] \\
&= A\cos\omega_0 t E[\sin\vartheta] + A\sin\omega_0 t E[\cos\vartheta] \\
&= A\cos\omega_0 t \int_{\vartheta=-\pi}^{\pi} \frac{1}{2\pi}\sin\vartheta\, d\vartheta + A\sin\omega_0 t \int_{\vartheta=-\pi}^{\pi} \frac{1}{2\pi}\cos\vartheta\, d\vartheta \\
&= \frac{A}{2\pi}\cos\omega_0 t \left[\cos\vartheta\right]_{\vartheta=-\pi}^{\pi} + \frac{A}{2\pi}\sin\omega_0 t \left[-\sin\vartheta\right]_{\vartheta=-\pi}^{\pi} \\
&= \frac{A}{2\pi}\cos\omega_0 t\left[\cos\pi - \cos(-\pi)\right] - \frac{A}{2\pi}\sin\omega_0 t\left[\sin\pi - \sin(-\pi)\right] \\
&= \frac{A}{2\pi}\cos\omega_0 t\left[1 - 1\right] - \frac{A}{2\pi}\sin\omega_0 t\left[0 - 0\right] \\
&= 0
\end{aligned}
$$

2.

$$
\begin{aligned}
E[X^2(t)] &= E[A^2\cos^2(\omega_0 t + \vartheta)] \\
&= A^2 E\left[\frac{1 + \cos 2(\omega_0 t + \vartheta)}{2}\right] \\
&= A^2 E\left[\frac{1}{2}\right] + \frac{A^2}{2}E\left[\cos 2(\omega_0 t + \vartheta)\right] \\
&= A^2 \cdot \frac{1}{2} + 0 \\
&= \frac{A^2}{2}
\end{aligned}
$$

3. $\sigma_X^2 = m_2 - m_1^2 = \frac{A^2}{2} - 0^2 = \frac{A^2}{2}$

4.

$$
\begin{aligned}
R_{XX}(\tau) &= E[X(t)X(t+\tau)] \\
&= \quad \text{Let } t = t_1; \quad t + \tau = t_2; \\
&= E[X(t_1)X(t_2)] \\
&= E\left[A\cos(\omega_0 t_1 + \vartheta)\cdot A\cos(\omega_0 t_2 + \vartheta)\right] \\
&= A^2 E\left[\cos(\omega_0 t_1 + \vartheta)\cdot\cos(\omega_0 t_2 + \vartheta)\right]
\end{aligned}
$$

$$= A^2 E\,[\cos(\omega_0 t_1 - \omega_0 t_2) + \cos(\omega_0 t_1 + \omega_0 t_2 + 2\vartheta)]$$

$$= \frac{A^2}{2} E\,[\cos(\omega_0 t_1 - \omega_0 t_2)] + \cancel{\frac{A^2}{2} E\,[\cos(\omega_0 t_1 + \omega_0 t_2 + 2\vartheta)]}^{=0}$$

$$= \frac{A^2}{2} E\,[\cos \omega_0 (t_1 - t_2)]$$

$$\frac{A^2}{2} E\,[\cos(\omega_0 \tau)]$$ ‘ϑ’ is a random variable

$$\therefore R_{XX}(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)$$

5.

$$C_{XX}(\tau) = E\left[\left(X(t) - \overline{X(t)}\right)\left(X(t+\tau) - \overline{X(t+\tau)}\right)\right]$$

$$= R_{XX}(\tau) - \left(\overline{X(t)}\right)^2$$

$$= \frac{A^2}{2} \cos(\omega_0 \tau) - 0^2$$

$$\therefore C_{XX}(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)$$

If $t_1 = t_2 = t$ then $C_{XX}(\tau) = \frac{A^2}{2}$

6.

$$E[Y(t)] = E[B \cos \omega_0 t]$$

$$= \cos \omega_0 t \; E[B]$$

$$= \cos \omega_0 t \times 0$$

$$\therefore E[Y(t)] = 0$$

Here r.v is ‘B’ given $F_B(b) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2}}$ By comparing their mean value $E[B] = 0$ and variance $\sigma_B^2 = 1$

$$\Rightarrow \sigma_B^2 = E[B^2] - (E[B])^2 \Rightarrow E[B^2] = 1$$

7.

$$E[Y^2(t)] = E[B^2 \cos^2 \omega_0 t]$$

$$= \cos^2 \omega_0 t \, E[B^2]$$

$$= \cos^2 \omega_0 t$$

8.

$$\sigma_Y^2 = E[Y^2(t)] - (E[Y(t)])^2$$

$$= \cos^2 \omega_0 t - 0$$

$$= \cos^2 \omega_0 t$$

9.

$$R_{YY}(\tau) = E[Y(t) Y(t+\tau)]$$

$$= E[B \cos \omega_0 t \; B \cos(\omega_0 t + \tau)]$$

$$= \cos \omega_0 t E[B^2]$$

$= \cos \omega_0 t$

10.

$$C_{YY}(\tau) = R_{YY}(\tau) - (\overline{Y})^2$$
$$= \cos \omega_0 t - 0$$
$$= \cos \omega_0 t$$

11.

$$R_{XY}(\tau) = E[A\cos(\omega_0 t_1 + \vartheta)\ B\cos(\omega_0 t_2 + \vartheta)]$$
$$= E[A\cos(\omega_0 t_1 + \vartheta)]E[B\cos(\omega_0 t_2 + +\vartheta)]$$
$$= 0$$

12.

$$C_{XY}(\tau) = R_{XY}(\tau) - (\overline{X}\,\overline{X}) = 0$$

$R_{XY}(t) = \overline{X}\,\overline{X}$ thus $X(t)$ and $Y(t)$ are uncorrelated.
Two process $X(t)$ and $Y(t)$ are called orthogonal then $E[X(t_1)X(t_2)] = 0$

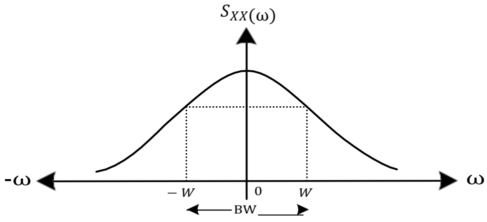
8.3 Types of random process

Two types:

- Baseband random process
- Bandpass random process

8.3.1 Baseband random process

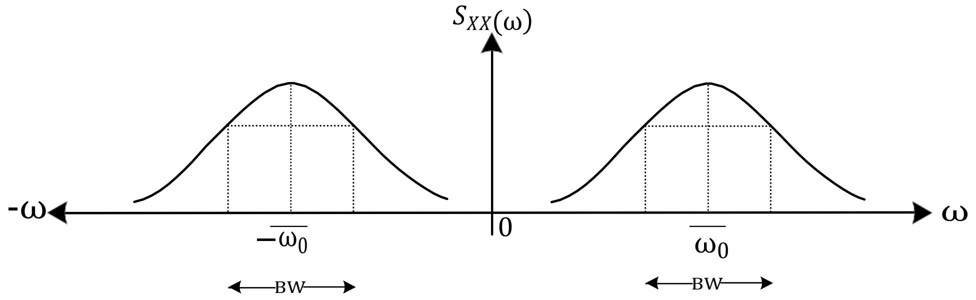
If the power spectral density $S_{XX}(\omega)$ of a random process $X(t)$have zero frequency components then it is called baseband random process. The frequency plot of baseband random process will be shown in Fig. Here 3dB bandwith can be written as



$$W_{rms} = \text{rms bandwidth} = \sqrt{\frac{\int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)d\omega}}$$

8.3.2 Bandpass random process

If the power spectral density $S_{XX}(\omega)$ of a random process $X(t)$ does not have zero frequency components then it is called bandpass random process. The frequency plot of baseband random process will be shown in Fig.



$$W_{rms} = \text{rms bandwidth} = \sqrt{\frac{4\int_{\omega=-\infty}^{\infty}\left(\omega-\overline{\omega_0}\right)^2 S_{XX}(\omega)d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)d\omega}}$$

$$\text{where } \overline{\omega_0} = \frac{\int_{\omega=0}^{\infty}\omega S_{XX}(\omega)d\omega}{\int_{\omega=0}^{\infty} S_{XX}(\omega)d\omega}$$

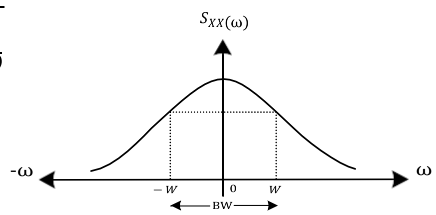
Question 1: What is the bandwidth of the power density spectrum?

Assume $X(t)$ is a lowpass random process,i.e its spectral components are clustered near $w = 0$ and have decreasing magnitude at higher frequencies.Except for the fact that the area of $S_{XX}(\omega)$ is not necessarily unity,$S_{XX}(\omega)$ has characteristics similar to probability density function(PDF).Indeed,by dividing $S_{XX}(\omega)$ by its area,a new function is formed with area of unity that is analogous to a density function.

Standard deviation is a measure of the spread in a density function.The analogous quantity for the normalized power spectrum is a measure of its spread that we call rms bandwidth ,which we denoted by W_{rms}.

Now since $S_{XX}(\omega)$ is an even function for a real process, its "mean value" is zero and its standard deviation is the square root of the its second moment, thus upon normalization, the rms bandwidth is given by

$$W_{rms} = \text{rms bandwidth} = \sqrt{\frac{\int_{\omega=-\infty}^{\infty}\omega^2 S_{XX}(\omega)d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)d\omega}}$$



It is also called Baseband random process.

Problem 9: The PSD of a baseband random process $X(t)$ is $S_{XX}(\omega) = \frac{2}{1+\left(\frac{\omega}{2}\right)^2}$

Find rms BW?

Solution:

$$W_{rms}^2 = \frac{\int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)d\omega}$$

Numerator part:

$$\int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega$$

$$= \int_{\omega=-\infty}^{\infty} \left[\frac{\omega^2 \times 2}{1+\left(\frac{\omega}{2}\right)^2}\right] d\omega$$

$$= \int_{\omega=-\infty}^{\infty} \frac{2\omega^2}{1+\frac{\omega^2}{4}} d\omega$$

$$= \int_{\omega=-\infty}^{\infty} \frac{2 \times 4\omega^2}{4+\omega^2} d\omega$$

$$= 2 \times 8 \int_{\omega=0}^{\infty} \frac{\omega^2}{4+\omega^2} d\omega$$

put $\omega = 2\tan\vartheta \Rightarrow d\omega = 2\sec^2\vartheta\, d\vartheta$

$$= 16 \int_{\omega=0}^{\infty} \frac{4\tan^2\vartheta}{4+4\tan^2\vartheta} \cdot 2\sec^2\vartheta$$

$$= 16 \times 2 \int_{\omega=0}^{\infty} \frac{4\tan^2\vartheta}{4(1+\tan^2\vartheta)} \sec^2\vartheta$$

$$= 32 \int_{\omega=0}^{\infty} \sin^2\vartheta \cos^{-2}\vartheta\, d\vartheta$$

$$= 32\, \beta\left(\frac{3}{2}, -\frac{1}{2}\right) \qquad \because \int_{0}^{\infty} \sin^m\vartheta \cos^n\vartheta\, d\vartheta = \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= 32 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)} \qquad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= 32 \frac{\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)}$$

$$= 32 \times \frac{1}{2}\,\frac{\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \qquad \Gamma(1+n) = n\Gamma(n)$$

$$= 32 \times \frac{1}{2}\sqrt{\pi} \times 2\sqrt{\pi} \qquad \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$= 32\pi$$

$$\boxed{\therefore \int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega = 32\pi}$$

Denominator part:

$$\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \int_{\omega=-\infty}^{\infty} \frac{2 \times 4}{2^2 + \omega^2}\, d\omega$$

$$= 8\left[\frac{1}{2}\tan^{-1}\frac{\omega}{2}\right]_{\omega=-\infty}^{\infty}$$

$$= 4\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

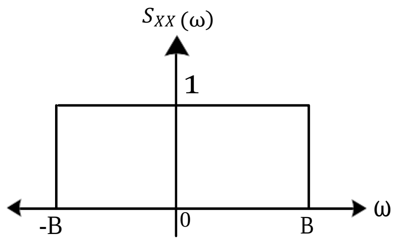
$$= 4\pi$$

$$W_{rms}^2 = \frac{\int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega} = \frac{32\pi}{4\pi} = 8$$

$$\Longrightarrow \boxed{\therefore W_{rms} = \sqrt{8}}$$

Problem 10: Assume random process PSD $S_XX(\omega) = \begin{cases} 1; & \text{for } |\omega| < B \\ 0; & \text{for } |\omega| \geq B \end{cases}$

Find the rms bandwidth?



Solution: Given

$$S_XX(\omega) = \begin{cases} 1; & -B \leq \omega \leq B \\ 0; & \text{otherwise} \end{cases}$$

$$W_{rms}^2 = \frac{\int_{\omega=-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$$

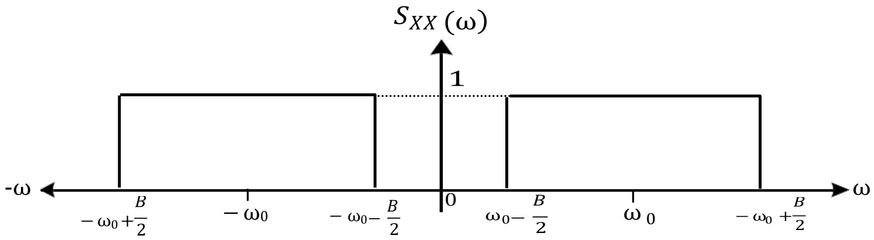
$$= \frac{\int_{\omega=-\infty}^{\infty} \omega^2 (1)\, d\omega}{\int_{\omega=-\infty}^{\infty} (1)\, d\omega} = \frac{\int_{\omega=-B}^{B} \omega^2\, d\omega}{\int_{\omega=-B}^{B} 1\, d\omega}$$

$$= \frac{\left[\frac{\omega^3}{3}\right]_{-B}^{B}}{\left[\omega\right]_{-B}^{B}} = \frac{\frac{2B^3}{3}}{2B}$$

$$= \frac{B^2}{3} \Longrightarrow \boxed{\therefore \text{ rms bandwidth } = W_{rms} = \frac{B}{\sqrt{3}}}$$

Problem 11: Assume random process PSD $S_{XX}(\omega) = \begin{cases} 1; & |\omega \pm \omega_0| < \frac{B}{2} \\ 0; & \text{elsewhere} \end{cases}$

Find the rms bandwidth?



Solution:

$$W_{rms}^2 = \frac{4\int_{\omega=-\infty}^{\infty} \left(\omega - \overline{\omega}_0\right)^2 S_{XX}(\omega)\, d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega}; \qquad \text{where} \quad \overline{\omega}_0 = \frac{\int_{\omega=0}^{\infty} \omega S_{XX}(\omega)\, d\omega}{\int_{\omega=0}^{\infty} S_{XX}(\omega)\, d\omega}$$

(i)

$$\int_0^{\infty} S_{XX}(\omega)\, d\omega = \int_{\omega_0 - \frac{B}{2}}^{\omega_0 + \frac{B}{2}} 1 \cdot d\omega = \left[\omega\right]_{\omega_0 - \frac{B}{2}}^{\omega_0 + \frac{B}{2}} = \omega_0 + \frac{B}{2} - \left(\omega_0 - \frac{B}{2}\right) = B$$

(ii)

$$\int_0^{\infty} \omega S_{XX}(\omega) = \int_{\omega_0 - \frac{B}{2}}^{\omega_0 + \frac{B}{2}} \omega \cdot 1 \cdot d\omega = \left[\frac{\omega^2}{2}\right]_{\omega_0 - \frac{B}{2}}^{\omega_0 + \frac{B}{2}}$$

$$= \frac{1}{2}\left\{\left(\omega_0 + \frac{B}{2}\right)^2 - \left(\omega_0 - \frac{B}{2}\right)^2\right\}$$

$$= \frac{1}{2} \times 4 \left(\omega_0 \cdot \frac{B}{2}\right) = B\omega_0$$

(iii)

$$\therefore \bar{\omega}_0 = \frac{\int_{\omega=0}^{\infty} \omega S_{XX}(\omega)d\omega}{\int_{\omega=0}^{\infty} S_{XX}(\omega)d\omega} = \frac{B\omega_0}{B} = B \qquad \because (i) \text{ and } (ii)$$

(iv)

$$W_{rms}^2 = \frac{4\int_{\omega=-\infty}^{\infty} \left(\omega - \bar{\omega}_0\right)^2 S_{XX}(\omega)d\omega}{\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)d\omega} \tag{8.4}$$

By taking numerator term $\bar{\omega}_0 = B$ and $S_{XX}(\omega) = 1$

$$\int_{\omega=-\infty}^{\infty} \left(\omega - \bar{\omega}_0\right)^2 S_{XX}(\omega)d\omega$$

$$= \int_{\omega=\omega_0-\frac{B}{2}}^{\omega_0+\frac{B}{2}} \left(\omega - \omega_0\right)^2 \cdot 1 \cdot d\omega$$

$$= \int_{\omega=\omega_0-\frac{B}{2}}^{\omega_0+\frac{B}{2}} \left(\omega^2 + \omega_0^2 - 2\omega\omega_0\right) d\omega$$

$$= \left[\frac{\omega^3}{3} + \omega_0^2\omega - 2\omega_0\frac{\omega^2}{2}\right]$$

$$= \frac{1}{3}\left\{\left(\omega_0 + \frac{B}{2}\right)^3 - \left(\omega_0 - \frac{B}{2}\right)^3\right\} + \omega_0^2\left[\left(\omega_0 + \frac{B}{2}\right) - \left(\omega_0 - \frac{B}{2}\right)\right]$$

$$- \omega_0\left\{\left(\omega_0 + \frac{B}{2}\right)^2 - \left(\omega_0 - \frac{B}{2}\right)^2\right\}$$

$\because (a+b)^3 - (a-b)^3 = 6a^2b + 2b^3; \qquad \because (a+b)^2 - (a-b)^2 = 4ab$

$$= \frac{1}{2}\left\{6\omega_0^2\frac{B}{2} + 2\left(\frac{B}{2}\right)^3\right\} + \omega_0^2 \cdot \frac{2B}{2} - \omega_0\left[4\omega_0\frac{B}{2}\right]$$

$$= \frac{1}{2}\left[6\omega_0^2\frac{B}{2} + \frac{B^3}{4}\right] + \omega_0^2 B - \frac{4\omega_0^2 B}{2}$$

$$= \omega_0^2 B + \frac{B^3}{12} + \omega_0^2 B - 2\omega_0^2 B$$

$$= 2\omega_0^2 B + \frac{B^3}{12} - 2\omega_0^2 B$$

$$= \frac{B^3}{12}$$

From the equation (8.4) and solution of (i) then

$$\Longrightarrow W_{rms}^2 = \frac{4 \cdot \frac{B^3}{12}}{B} = \frac{B^2}{3}$$

$$\Longrightarrow \boxed{\therefore W_{rms} = \text{rms bandwidth} = \frac{B}{\sqrt{3}}}$$

- Both, the ideal low pass and band pass process, rms bandwidth is equal i.e., $\frac{B}{\sqrt{3}}$. This is the only the case if the factor is present 4 in bandwidth of band-pass random process.

8.4 Cross correlation and cross PSD

Let two random process $X(t)$ and $Y(t)$, the sample function of random processcan be written as

$$X_T(t) = \begin{cases} X(t); & -T \le t \le T \\ 0; & \text{elsewhere} \end{cases}$$

$$Y_T(t) = \begin{cases} y(t); & -T \le t \le T \\ 0; & \text{elsewhere} \end{cases}$$

The Fourier Transform of $X_T(t)$ can be written as

$$F\ X(t) = X_T(\omega) = \int_{t=-\infty}^{\infty} X_T(t) e^{-j\omega t}\, dt$$

$$= \int_{t=-T}^{T} X_T(t) e^{-j\omega t}\, dt$$

$$\therefore X_T(\omega) = \int_{t=-T}^{T} X(t) e^{-j\omega t}\, dt$$

The Fourier Transform of $Y_T(t)$ can be written as

$$F\ X(t) = Y_T(\omega) = \int_{t=-\infty}^{\infty} Y_T(t) e^{-j\omega t}\, dt$$

$$= \int_{t=-T}^{T} Y_T(t) e^{-j\omega t}\, dt$$

$$\therefore Y_T(\omega) = \int_{t=-T}^{T} Y(t) e^{-j\omega t}\, dt$$

The cross-power between $X(t)$ and $Y(t)$ in interval $(-T, T)$ can be written as

$$P_{XY} = \int_{t=-\infty}^{\infty} X_T(t)Y_T(t)\, dt = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \frac{X_T^*(\omega)Y_T(\omega)}{2T}\, d\omega$$

The total cross-power can be written as

$$\lim_{T\to\infty} \frac{1}{2T}\int_{t=-\infty}^{\infty} X_T(t)Y_T(t)\, dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \frac{X_T^*(\omega)Y_T(\omega)}{2T}\, d\omega$$

The total average cross power can be written as

$$\therefore P_{XY} = \lim_{T\to\infty} \frac{1}{2T}\int_{t=-\infty}^{\infty} E[X_T(t)Y_T(t)]\, dt = \lim_{T\to\infty} \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T}\, d\omega$$

$$\therefore P_{XY} = A \quad E[X_(t)Y_T(t)] \quad = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XY}(\omega)\, d\omega$$

where S_{XY} is cross PSD can be written as

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)Y_T(\omega)]}{2T}\, d\omega$$

8.4.1 Wiener Kinchin Relation

The Wiener Kinchin relation says that Cross Power Spectral Density (PSD) $S_{XY}(\omega)$ and Cross-correlation function $R_{XY}(\tau)$ from the Fourier Transform pair.

$$S_{XY}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\, d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XY}(\omega)e^{+j\omega\tau}\, d\omega$$

$$\therefore R_{XY}(\tau) \overset{F}{\longleftrightarrow} S_{XY}(\omega)$$

Proof. Let $X(t)$ be the random process with PSD of

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left\{\int_{t=-T}^{T} X_T(t)e^{j\omega t}\, dt \cdot \int_{t=-T}^{T} Y_T(t)e^{-j\omega t}\, dt\right\}$$

Where $X_T(t)$ is obtained from random random process $X(t)$ as $t = t_1$ and $Y(t)$ is obtained $t = t_2 = t_1 + \tau$ then

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left\{\int_{t_1=-T}^{T} X_T(t_1)e^{j\omega t_1} dt_1 \cdot \int_{t_2=-T}^{T} Y_T(t_2)e^{-j\omega t_2}\, dt_2\right\}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{t_1=-T}^{T} \int_{t_2=-T}^{T} E\left(X_T(t_1)Y_T(t_2) \right) e^{-j\omega(t_2-t_1)}\, dt_2\, dt_1$$

let $t_1 = T, t_2 = t_1 + \tau \Rightarrow \tau = t_2 - t_1$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} \int_{\tau=T-t}^{+t} E\left(X_T(t)Y_T(t+\tau) \right) e^{-j\omega\tau}\, d\tau\, dt$$

$$= \int_{\tau=-T}^{T} \left[\lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} R_{XY}(\tau) e^{-j\omega\tau}\, dt \right] d\tau$$

$$= \int_{\tau=-T}^{T} A\,[R_{XY}(\tau)]\, e^{-j\omega\tau}\, d\tau$$

The random process $X(t)$ and $Y(t)$ are WSS random process, then

$$S_{XY}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau = F\left(R_{XY}(\tau) \right)$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega = F^{-1}\left(S_{XY}(\omega) \right)$$

□

8.4.2 Properties of Cross Power Spectral Density (PSD)

1. Power spectal density is even function, $S_{XY}(\omega) = S_{XY}(-\omega)$

   Proof.

$$S_{XY}(-\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau}\, d\tau \tag{8.5}$$

Let $\omega = -\omega$

$$S_{XY}(-\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau) e^{+j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \underbrace{R_{XY}(\tau)}_{\text{even}} e^{-j\omega(-\tau)}\, d\tau$$

Cross-correlation function is even function $R_{XY}(\tau) = R_{XY}(-\tau)$

$$S_{XY}(-\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(-\tau) e^{-j\omega(-\tau)}\, d\tau \qquad (8.6)$$

From equation (8.5) and (8.6)

$$\boxed{S_{XY}(-\omega) = S_{XY}(\omega)}$$

□

2. The real part of cross PSD is even and imaginary part is odd function.

Proof.

$$S_{XY}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} R_{XY}(\tau)\, [\cos\omega\tau - j\sin\omega\tau]\, d\tau$$

$$Re\,[S_{XY}(\omega)] = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau) \cos\omega\tau\, d\tau \implies \text{even function}$$

$$Im\,[S_{XY}(\omega)] = -\int_{\tau=-\infty}^{\infty} R_{XY}(\tau) \sin\omega\tau\, d\tau \implies \text{odd function}$$

□

3. If $X(t)$ and $Y(t)$ are orthogonal randomprocess then cross PSD is zero.

Proof.

$$S_{XY}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau}\, d\tau$$

$$R_{XY}(\tau) = E\,[X(t)X(t+\tau)] = 0$$ If $X(t)$ and $Y(t)$ are orthogonal.

$$\boxed{\therefore S_{XY}(\omega) = 0}$$

□

4. If $X(t)$ and $Y(t)$ are uncorrelated and WSS r.p then $S_{XY} = 2\pi \overline{X}\,\overline{Y}\,\delta(\omega)$

Proof. From Wiener Kinchin relation

$$S_{XX}(\omega) = \int_{\tau=-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} E\,[X(t)Y(t+\tau)]\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} E\,[X(t)]\, E\,[Y(t+\tau)]\, e^{-j\omega\tau}\, d\tau \qquad \because X(t),\, Y(t) \text{ are independent}$$

$$= \int_{\tau=-\infty}^{\infty} \overline{X}\,\overline{Y} e^{-j\omega\tau}\, d\tau \qquad \because \text{WSS } E[X(t)] = \overline{X};\ E[Y(t+\tau)] = \overline{Y}$$

$$= \overline{X}\,\overline{Y} \int_{\tau=-\infty}^{\infty} e^{-j\omega\tau}\, d\tau$$

$$= \overline{X}\,\overline{Y} \cdot 2\pi\delta(\omega)$$

$$\boxed{\therefore\ S_{XX}(\omega) = 2\pi\overline{X}\,\overline{Y}\delta(\omega)}$$

□

8.5 White Noise

A white noise in which all frequency components from $f = -\infty$ to $f = \infty$ are present in equal measure i.e., whose PSD remains constant for all frequencies and is independent of frequency, which is called "white noise". It is shown in figure.

$$\therefore\ S_N(f) = \frac{N_0}{2}; \quad -\infty \le f \le \infty; \qquad N_0 \text{ is constant}$$

∴ The white noise process is zero mean WSS process with PSD is constant (flat) for all frequencies. It is strictly speaking if we take inverse fourier transform of a flat function does not exist for all frequencies f.

$$F^{-1}\{S_N(f)\} = R_{WW}(f) = F^{-1}\frac{N_0}{2} = \frac{N_0}{2}\delta(\tau)$$

But from definition,

$$R_{WW}(\tau) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_W(f)\,d\omega = \int_{\omega=-\infty}^{\infty} S_W(f)\,df$$

If $\tau = 0$, we will get mean square value, so,

$$R_{WW}(\tau) = \overline{W^2} = \int_{\omega=-\infty}^{\infty} \frac{N_0}{2}\, df = \infty$$

So, this mean square value (power) of white process is infinite. However it is not possible to have a random process with infinite power, white noise does not exist in the physical world. It is mathematical model can be used a close approximation toreal world process.

Gaussian white noise often called white Gaussian noise, for any two (or several) random variables from the process independent and long as they are not same random variable, and uncorrelated their mean is zero.

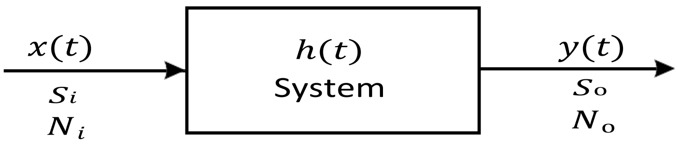
8.6 Signal to Noise Ratio (SNR)

The SNR is defined as

$$SNR = \frac{\text{Signal power}}{\text{Noise power}}$$

Here SNR is a ratio of powers and not of voltages. We may express SNR is in decibels rather than just a ratio.

$$(SNR)_{dB} = 10 \log_{10} \frac{\text{Signal power}}{\text{Noise power}}$$



Here,

$S_i -$ input signal power $\quad S_o -$ input signal power

$N_i -$ input signal power $\quad N_o -$ input signal power

$G -$ system gain

The output signal power $S_0 = GS_i$

and output noise power $N_0 = GN_i + N_a$

where $N_a -$ is additional noise power in the system.

Input SNR $\left(\frac{S}{N}\right)_i = \frac{\text{Input signal power}}{\text{Input noise power}} = \frac{S_i}{N_i}$

Output SNR $\left(\frac{S}{N}\right)_o = \frac{\text{Output signal power}}{\text{Output noise power}} = \frac{GS_i}{GN_i + N_a}$

The output $(SNR)_{10}$ < Input $(SNR)_{10}$

$$\therefore\ SNR = \frac{(S/N)_o}{(S/N)_i}$$

For any circuit, contain some noise producing active/ passive elements in it. These SNR at the output will always be less than the SNR at the input, i.e., there is a deterioration of SNR. Thus an amplifier does not improve SNR, it only degrades it.

Noise figure:

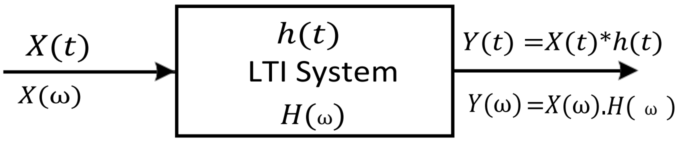
$$F = \text{Noise figure} = \frac{(S/N)_i}{(S/N)_o}$$

CHAPTER 9

LTI Systemswith Random Inputs

9.1 Introduction

In application of random process, the input-output relation through linear system can be described as follows.



Here $X(t)$ is a random process and $h(t)$ (deterministic function) is the impulse response of the linear system (Filter or another linear system).

9.1.1 Input-Output relation

1. Time domain: The output is the time domain is convolution of the input random process $X(t)$ and impulse response $h(t)$ i.e.,

$$y(t) = X(t) * h(t) = \int_{\tau=-\infty}^{\infty} X(\tau)h(t-\tau)\, d\tau = \int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau$$

   Q: Can you evaluate this convolution integral?

   A: In general, we can not evaluate this convolution integral, because $X(t)$ is random process and there is no mathematical expression for $X(t)$.

2. Frequency domain: The output in the frequency domain is the product of the input Fourier transform of the impulse response $X(t)$ is $X(f)$ and the fourier transform of the impulse response $h(t)$ is $H(f)$.

$$X(f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}\, dt \implies \text{FT of the the input r.p } X(t) \text{ is a r.p}$$

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\, dt \implies \text{FT of the the deterministic impulse response}$$

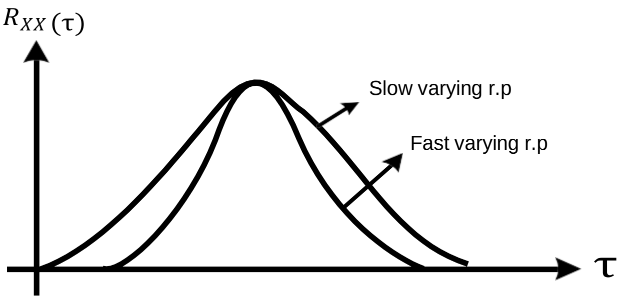
$$\boxed{Y(f) = X(f)H(f) \quad \text{or} \quad Y(\omega) = X(\omega)H(\omega)}$$

Q: Can you evaluate the Fourier transform of input random porocess $x(t)$, $X(f)$?
A: In genertal NO. Since, the $X(t)$ is random in general and has no mathematical expression.

Q: How can we describe the behavior of the random process and the output random process through a linear system?
A: Case 1: Using auto-correlation function of random process $X(t)$, $R_{XX}(\tau)$. assume a WSS (constant mean and $R_{XX}(\tau)$ function is deterministic and only a function of 'τ'.

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

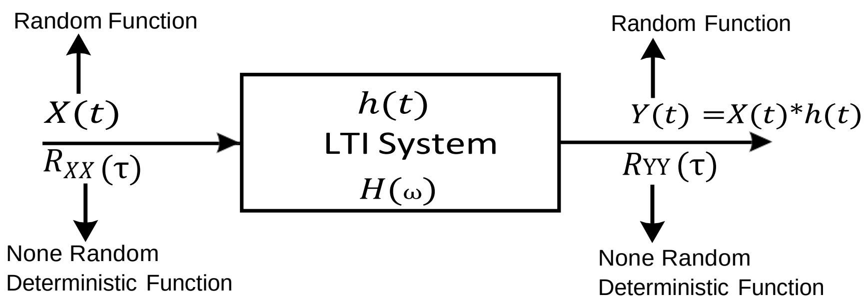
The auto-correlation tell us how the random process is varying. It is a slow varying/fast varying process.



Case 2: Using power spectral density.

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

NOTE: The inout and output of the linear system as shown below in time and frequency domain assuming the random process $X(t)$ is WSS.



9.1.2 Response of LTI system in time domain

1. Response of LTI system for mean value
   Let $X(t)$ is a random process with mean $E[X(t)]$ and the response of the system

for mean value can be written as

$$y(t) = X(t) * h(t) = \int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau$$

Take expectation on both sides,

$$E[Y(t)] = E\left[\int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau\right] = \int_{\tau=-\infty}^{\infty} h(\tau)E\{X(t-\tau)\}\, d\tau$$

If $X(t)$ be the WSS process, $E[X(t)] = E[X(t+\tau)]$

$$\therefore\ E[Y(t)] == \overline{X(t)} \int_{\tau=-\infty}^{\infty} h(\tau)\, d\tau$$

The mean value of $Y(t)$ is the multiplication of mean value of $X(t)$ and the area under the impulse response.

2. Response of LTI system for mean-square value

   Let the output of LTI system,

$$y(t) = X(t) * h(t) = \int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau$$

The mean-square value can be written as

$$E[Y^2(t)] = E\left[\int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau\right]^2$$

$$= E\left[\int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau \cdot \int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau\right]$$

$$= E\left[\int_{\tau_1=-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\, d\tau_1 \cdot \int_{\tau_2=-\infty}^{\infty} h(\tau_2)X(t-\tau_2)\, d\tau_2\right]$$

$$= \int_{\tau_1=-\infty}^{\infty}\int_{\tau_2=-\infty}^{\infty} E[X(t-\tau_1)\ X(t-\tau_2)]h(\tau_1)h(\tau_2)\, d\tau_1\, d\tau_1$$

$$= \int_{\tau_1=-\infty}^{\infty}\int_{\tau_2=-\infty}^{\infty} R_{XX}(\tau_1 - \tau_2)h(\tau_1)h(\tau_2)\, d\tau_1\, d\tau_2$$

Let $\tau_1 = \tau_2 = \tau$

$$\boxed{E[Y^2(t)] = \int_{\tau=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} R_{XX}(0)h(\tau)h(\tau)\ d\tau\ d\tau}$$

3. Response of LTI system for auto-correlation function (ACF)

   Let the output function $Y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\ d\tau$

   The auto-correlation of $X(t)$ and $Y(t)$ is

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\left\{\underbrace{\int_{\tau_1=-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\ d\tau_1}_{Y(t)} \cdot \underbrace{\int_{\tau_2=-\infty}^{\infty} h(\tau_2)X(t-\tau_2)\ d\tau_2}_{Y(t+\tau)}\right\}$$

$$= \int_{\tau_1=-\infty}^{\infty}\int_{\tau_2=-\infty}^{\infty} E[X(t-\tau_1)X(t+\tau-\tau_2)]\ h(\tau_1)h(\tau_2)\ d\tau_1\ d\tau_2$$

$$= \int_{\tau_1=-\infty}^{\infty}\int_{\tau_2=-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\ h(\tau_1)h(\tau_2)\ d\tau_1\ d\tau_2$$

$$\boxed{\therefore\ R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)}$$

Other method:

$$R_{YY}(\tau) = E\left\{\int_{\tau=-\infty}^{\infty} h(\tau)X(t-\tau)\ d\tau \cdot \int_{\tau=-\infty}^{\infty} h(t+\tau)X(t)\ d\tau\right\}$$

$$= \int_{\tau=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} E[X(t)X(t-\tau)]\ h(\tau)h(t+\tau)\ d\tau\ d\tau$$

$$\boxed{\therefore\ R_{YY}(\tau) = \int_{\tau=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} R_{XX}(\tau)h(\tau)h(t+\tau)\ d\tau\ d\tau}$$

4. Cross-correlation function of input and output

   The cross-correlation of $X(t)$ and $Y(t)$ is

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

$$= E\left\{X(t)\int_{\tau_1=-\infty}^{\infty} h(\tau_1)X(t+\tau-\tau_1)\ d\tau_1\right\}$$

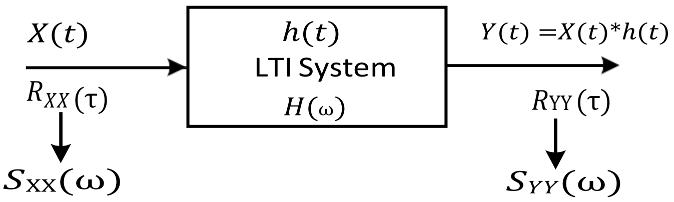
$$= E\ [X(t)X(t+\tau-\tau_1)]\ h(\tau_1)\ d\tau_1$$

$$= R_{XX}(\tau - \tau_1)h(\tau_!)\ d\tau_1$$

$$\boxed{\therefore\ R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)}$$

9.1.3 Response of LTI system in frequency domain

1. Response of LTI systen for PSD



The output PSD $S_{YY}(\omega) = F\ R_{YY}(\tau) = \int_{\tau=-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}\, d\tau$ (9.1)

ACF $R_{YY}(\omega) = E\ Y(t)Y(t+\tau)$ (9.2)

$$Y(t) = X(t) * h(t) = \int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)X(t-\alpha_1)\ d\alpha_1$$

$$Y(t+\tau) = X(t+\tau) * h(t+\tau) = \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)X(t+\tau-\alpha_2)\ d\alpha_2$$

From equation (9.2)

$$R_{YY}(\tau) = E\left\{\int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)X(t-\alpha_1)\ d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)X(t+\tau-\alpha_2)\ d\alpha_2\right\}$$

$$= \int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)\, d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)\ d\alpha_2 \cdot E[X(t-\alpha_1)X([t+\tau]-\alpha_2)]$$

let $T = t - \alpha_1$; $\quad T + \tau = t + \tau - \alpha_2$

$$T + \tau - T \Rightarrow (t+\tau) - \alpha_2 - (t - \alpha_1) = \tau + \alpha_1 - \alpha_2$$

$$R_{YY}(\tau) = \int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)\, d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)\ d\alpha_2 \cdot R_{XX}(\tau + \alpha_1 - \alpha_2)$$

From equation (9.1)

$$S_{YY}(\omega) = \int_{\tau=-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}\ d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \left[\int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)\, d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)\, d\alpha_2 \cdot R_{XX}(\tau + \alpha_1 - \alpha_2) e^{-j\omega\tau}\, d\tau \right]$$

let $\alpha = \tau + \alpha_1 - \alpha_2$

$$= \int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)\, d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)\, d\alpha_2 \cdot \int_{\tau=-\infty}^{\infty} R_{XX}(\alpha) e^{-j\omega(\alpha - \alpha_1 + \alpha_2)}\, d\alpha$$

$$= \int_{\alpha_1=-\infty}^{\infty} h(\alpha_1)\, d\alpha_1 \cdot \int_{\alpha_2=-\infty}^{\infty} h(\alpha_2)\, d\alpha_2 \cdot \int_{\alpha=-\infty}^{\infty} R_{XX}(\alpha) e^{-j\omega(\alpha)}\, d\alpha$$

$$= H(-\omega) \cdot H(\omega) \cdot S_{XX}(\omega)$$

$$= \left|H(\omega)\right|^2 \cdot S_{XX}(\omega)$$

$$\boxed{\therefore\ S_{YY}(\omega) = S_{XX}(\omega) \cdot \left|H(\omega)\right|^2}$$

Alternate Method:

From response of LTI system for ACF

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot H^*(\omega) \cdot H(\omega)$$

$$\boxed{\therefore\ S_{YY}(\omega) = S_{XX}(\omega) \cdot \left|H(\tau)\right|^2}$$

2. Power calculation at input and output of LTI system

Total power of the input

$$P_{XX} = E[X^2(t)] = R_{XX}(0) = \int_{f=-\infty}^{\infty} S_{XX}(f)\, df$$

Total power of the output

$$P_{YY} = E[Y^2(t)] = R_{YY}(0) = \int_{f=-\infty}^{\infty} S_{YY}(f)\, df = \int_{-\infty}^{\infty} S_{XX}(f) \left|H(f)\right|^2 df$$

Problem 1: Let $X(t)$ be the random process with PSD $S_{XX}(\omega)$ is shown in Fig. Find the output power of a LTI system whose frequency response is

$$H(\omega) = \begin{cases} 1; & |\omega| \le \omega_c \\ 0; & \text{otherwise} \end{cases}$$

Solution: (i) Average power in-terms of angular frequency

$$P_{YY} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega = \frac{1}{2\pi}\int_{\omega=-\omega_c}^{\omega_c} \frac{\eta}{2}\, d\omega$$

$$= \frac{1}{2\pi}\times\frac{\eta}{2}\left[\omega\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\times\frac{\eta}{2}\, 2\omega_c$$

$$= \frac{\eta\omega_c}{2\pi}\ Watts/rad/sec \quad or \quad V^2/rad/sec$$

(ii) Average power in-terms of linear frequency

$$P_{YY} = \int_{f=-\infty}^{\infty} S_{XX}(f)\, df = \int_{f=-f_c}^{f_c} \frac{\eta}{2}\, df$$

$$= \frac{\eta}{2}\left[f\right]_{-f_c}^{f_c} = \frac{\eta}{2}\, 2f_c$$

$$= \eta f_c\ Watts/Hz \quad or \quad V^2/Hz$$

Problem 2: Let $X(t)$ be the random process with PSD $S_{XX}(\omega)$ is shown in Fig. Find the output power of a LTI system whose frequency response is

$$H(\omega) = \begin{cases} 1; & (\omega_c - \frac{B}{2}) \le |\omega| \le (\omega_c + \frac{B}{2}) \\ 0; & \text{otherwise} \end{cases}$$

Solution: (i) Average power in-terms of angular frequency

$$P_{YY} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\times 2\int_{\omega=\omega_c-\frac{B}{2}}^{\omega_c+\frac{B}{2}} \frac{\eta}{2}\, d\omega$$

$$= \frac{1}{2\pi}\times 2\times\frac{\eta}{2}\Big[\omega\Big]_{\omega_c-\frac{B}{2}}^{\omega_c+\frac{B}{2}}$$

$$= \frac{1}{2\pi}\times\frac{\eta}{2}\left[\omega_c+\frac{B}{2}-\left(\omega_c-\frac{B}{2}\right)\right]$$

$$= \frac{\eta B}{2\pi}\ Watts/rad/sec \quad or \quad V^2/rad/sec$$

(ii) Average power in-terms of linear frequency

$$P_{YY} = \frac{1}{2\pi}\int_{f=-\infty}^{\infty} S_{XX}(f)\, df$$

$$= 2\int_{f=f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} \frac{\eta}{2}\, df$$

$$= 2 \times \frac{\eta}{2} \left[f \right]_{f_c - \frac{B}{2}}^{f_c + \frac{B}{2}}$$

$$= \frac{\eta}{2} \left[f_c + \frac{B}{2} - \left(f_c - \frac{B}{2} \right) \right]$$

$$= \eta B \ Watts/Hz \ or \ V^2/Hz$$

Problem 3: A random noise $X(t)$ having power spectrum $S_{XX}(\omega) = \frac{3}{49 + \omega^2}$ is applied to a network for which $h(t) = t^2 Exp(-7t)$. The network response is denoted by $Y(t)$.

1. Find the average power of $X(t)$
2. Find the power spectrum of $Y(t)$
3. Find the average power of $Y(t)$

Solution:

$$P_{YY} = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} S_{XX}(\omega) \ d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \frac{3}{49 + \omega^2} d\omega$$

$$= \frac{3}{2\pi} \times \frac{1}{7} \tan^{-1} \left[\frac{\omega}{7} \right]_{\omega=-\infty}^{\infty}$$

$$= \frac{3}{2\pi} \times \frac{1}{7} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]$$

$$= \frac{3}{2\pi} \times \frac{1}{7} \times \pi$$

$$= \frac{3}{14} \ Watts$$

$$\therefore P_{XX} = \frac{3}{14} \ Watts$$

(ii) Given $h(t) = t^2 Exp(-7t)$

$$H(\omega) = F[h(t)] = \int_{\omega=-\infty}^{\infty} t^2 Exp(-7t) \cdot e^{-j\omega t} \ dt$$

$$= \int_{\omega=-\infty}^{\infty} t^2 e^{-(7+j\omega)t} \ dt$$

let $x = (7 + j\omega)t \Rightarrow t = \frac{x}{7 + j\omega}$;

$$dx = (7 + j\omega)\, dt \implies dt = \frac{dx}{7 + j\omega}$$

if $x = \infty \Rightarrow t = \infty$; $\quad x = -\infty \Rightarrow t = -\infty$

$$\therefore H(\omega) = \int_{x=-\infty}^{\infty} \frac{x^2}{(7 + j\omega)^2} \cdot e^{-x} \cdot \frac{dx}{7 + j\omega}$$

$$= \frac{1}{(7 + j\omega)^3} \int_{x=-\infty}^{\infty} e^{-x} x^2 \, dx \qquad \because \int_{x=-\infty}^{\infty} e^{-x} x^{n-1} \, dx = \Gamma(n)$$

$$= \frac{1}{(7 + j\omega)^3} \int_{x=-\infty}^{\infty} e^{-x} x^{3-1} \, dx \qquad \because \Gamma(n) = n\Gamma(n-1)$$

$$= \frac{1}{(7 + j\omega)^3} \Gamma(3) \qquad \because \Gamma(n+1) = n\Gamma(n) = n!$$

$$= \frac{1}{(7 + j\omega)^3} \Gamma(2 + 1)$$

$$\frac{1}{(7 + j\omega)^3} \times 2!$$

$$\therefore H(\omega) = \frac{2}{(7 + j\omega)^3}$$

$$|H(\omega)|^2 = \left|\frac{2}{(7 + j\omega)^3}\right|^2$$

$$= \frac{4}{\left((49 + \omega^2)^{\frac{3}{2}}\right)^2}$$

$$= \frac{4}{(49 + \omega^2)^3}$$

$$\therefore S_{YY}(\omega) = S_{XX}(\omega) \cdot |H(\omega)|^2$$

$$= \frac{3}{49 + \omega^2} \cdot \frac{4}{(49 + \omega^2)^3}$$

$$= \frac{12}{(49 + \omega^2)^4}$$

$$\boxed{\therefore S_{YY}(\omega) = \frac{12}{(49 + \omega^2)^4}}$$

(iii)

$$P_{YY}(\omega) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} S_{XX}(\omega)$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \frac{12}{(49 + \omega^2)^4} \, d\omega$$

let $\omega = 7\tan\vartheta \Longrightarrow d\omega = 7\sec^2\vartheta$

let $\omega = -\infty \Rightarrow \vartheta = \tan^{-1}(-\infty) = -\frac{\pi}{2}$;

$\omega = \infty \Rightarrow \vartheta = \tan^{-1}(\infty) = \frac{\pi}{2}$

$$P_{YY} = \frac{1}{2\pi}\int_{\omega=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{12}{(1+\tan^2\vartheta)^4}\cdot 7\sec^2\vartheta\, d\vartheta$$

$$= \frac{1}{2\pi}\int_{\omega=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{12}{(\sec^2\vartheta)^4}\cdot 7\sec^2\vartheta\, d\vartheta$$

$$= \frac{12\times 7}{2\pi}\int_{\omega=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sec^6\vartheta}\, d\vartheta$$

$$= \frac{12\times 7\times 2}{2\pi}\int_{\omega=0}^{\frac{\pi}{2}} \cos^6\vartheta\, d\vartheta \Rightarrow (1)$$

We know that

$$\int_0^{\infty} \cos^n\vartheta\, d\vartheta = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\frac{1}{2}\cdot\frac{\pi}{2}; & \text{'n' even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\ldots\frac{2}{3}; & \text{'n' odd} \end{cases}$$

$$\int_0^{\pi/2} \cos^6\vartheta\, d\vartheta = \frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\cdot\frac{\pi}{2}$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$(1) \Rightarrow P_{YY}(\omega) = \frac{12\times 7}{2\pi}\times\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{7\times 5\times 3}{2\times 4} = \frac{105}{8} = 13.125\ Watts$$

$$\therefore P_{YY} = 13.125\ Watts$$

Problem: 4 A random voltage modeled by a white noise process $X(t)$ which power spectral density $\frac{n}{2}$ ia an input to RC network shown in Fig. Find

1. Output PSD $S_{YY}(\omega)$
2. Auto-correlation function $R_{YY}(\tau)$
3. Average output power $[Y^2(t)]$

Solution: The frequency response of the system is given by

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

(a)

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{1}{1^2 + \omega^2 R^2 C^2} \times \frac{N_0}{2}$$

(b) Taking inverse Fourier transform both sides

$$R_{YY}(\tau) = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$

(c) Average output power

$$E[Y^2(t)] = R_{YY}(0) = \frac{N_0}{4RC}$$

Problem 5: A WSS r.p $X(t)$ with PSD $S_{XX}(f) = \begin{cases} 10^{-4}; & |f| < 100 \\ 0; & \text{otherwise} \end{cases}$ is the input an RC filter with the frequency response $H(f) = \frac{1}{100\pi + j2\pi f}$. The filter output is the stochastic process $Y(t)$. What is the

(a) $E[X^2(t)]$ (b) $S_{XY}(f)$ (c) $S_{YX}(f)$ (d) $S_{YY}(f)$ (e) $E[Y^2(t)]$

Solution:

(a) We know that

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\, d\omega = \int_{-\infty}^{\infty} S_{XX}(f)e^{j2\pi f\tau}\, df$$

If $\tau = 0$ $\quad R_{XX}(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(f)e^0\, df$

Mean square value

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(f)\, df$$

$$= \int_{-100}^{100} 10^{-4}\, df = 10^{-4}\left[f\right]_{-100}^{100} = 10^{-4}(200) = 0.02$$

$$\boxed{\therefore\ E[X^2(t)] = 0.02}$$

(b) $S_{XY}(f) = H(f)S_{XX}(f) = \begin{cases} 10^{-4}H(f); & |f| \le 100 \\ 0; & \text{otherwise} \end{cases}$

$$\therefore S_{XY}(f) = \begin{cases} \frac{10^{-4}}{100\pi + j2\pi f}; & |f| \le 100 \\ 0; & \text{otherwise} \end{cases}$$

(c) $S_{YX}(f) = S^*_{XY}(f)$ and we know that $R_{YX}(\tau) = R^*_{XY}(-\tau)$

$$S_{YX}(f) = S^*_{XY}(f) = \begin{cases} 10^{-4}H^*(f); & |f| \le 100 \\ 0; & \text{otherwise} \end{cases}$$

$$\therefore S_{YX}(f) = \begin{cases} \frac{10^{-4}}{100\pi - j2\pi f}; & |f| \le 100 \\ 0; & \text{otherwise} \end{cases}$$

(d) $S_{YY}(f) = H^*(f)S_{XY}(f) = |H(f)|^2 S_{XX}(f)$

$$\therefore S_{YX}(f) = \begin{cases} \frac{10^{-4}}{10^4\pi^2 + (2\pi f)^2}; & |f| \le 100 \\ 0; & \text{otherwise} \end{cases}$$

(e)

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_{YY}(f)\, df$$

$$= \int_{-100}^{100} \frac{10^{-4}}{10^4\pi^2 + 4\pi^2 f^2} df$$

$$= \frac{2}{10^8\pi^2} \int_0^{100} \frac{df}{1 + \left(\frac{f}{50}\right)^2}$$

$$= \frac{2}{10^8\pi^2} \left[\tan^{-1}\frac{f}{50}\right]_0^{100} \qquad \because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{2}{10^8\pi^2}\left[\tan^{-1}(2) - \tan^{-1}(0)\right]$$

$$= \frac{2}{10^8\pi^2} \times 63.4349$$

$$= 12.584 \times 10^8$$

$$\therefore E[Y^2(t)] = 12.584 \times 10^8$$

Problem 6: Let $X(t)$ is a WSS Gaussian r.p with mean $\overline{X(t)} = 0$ and auto-correlation function $R_{XX}(\tau) = 10\delta(\tau)$. where $\delta(\cdot)$ is a Dirac delta function. The random process $X(t)$ is run through a filter combination as shown below, where the first filter frequency response

$$H_1(f) = \begin{cases} 1; & |f| < 3 \\ 0; & \text{otherwise} \end{cases}$$

and second filter has frequency response $H_2(f) = e^{-2f^2}$

(a) Find the power $E[Y^2(t)]$ in $Y(t)$

(b) Find the power $E[Z^2(t)]$ in $Z(t)$

Solution: Given $R_{XX}(\tau) = 10\delta(\tau)$

We know that $R_{XX}(\tau) \leftrightarrow S_{XX}(f)$

$\therefore S_{XX}(f) = 10$

(a)

$$H_1(f) = \begin{cases} 1; & |f| < 3 \\ 0; & \text{otherwise} \end{cases}$$

$$Y(f) = |H_1(f)|^2 S_{XX}(f)$$

$$S_{YY}(f) = \begin{cases} 10; & |f| < 3 \\ 0; & \text{otherwise} \end{cases}$$

$$\therefore P_{YY} = \overline{Y^2(t)} = \int_{-\infty}^{\infty} S_{YY}(f)\, df = \int_{-3}^{3} 10\, df = 60\ Watts/Hz$$

(b) $H_2(f) = e^{-2f^2} \implies |H_2(f)|^2 = e^{-4f^4} \approx e^{-\frac{f^2}{2}}$

Because square of Gaussian r.v will become Gaussian r.v. So, $|H_2(f)|^2 = e^{-\frac{f^2}{2}}$ (approximation)

$$\therefore P_{ZZ} = E[Z^2(t)] = \int_{-\infty}^{\infty} |H_2(f)|^2 S_{YY}(f)\, df$$

$$S_{ZZ}(f) = \begin{cases} e^{-\frac{f^2}{2}}; & |f| < 3 \\ 0; & \text{otherwise} \end{cases}$$

$$\therefore P_{ZZ} = E[Z^2(t)] = \int_{-\infty}^{\infty} S_{ZZ}(f)\, df$$

$$= \int_{-3}^{3} 10 e^{-\frac{f^2}{2}}\, df$$

$$= 10\sqrt{2\pi}\int_{-3}^{3}\frac{1}{\sqrt{2\pi}}e^{-\frac{f^2}{2}}\,df$$
$$= 10\sqrt{2\pi}[2-2Q(3)]$$
$$= 10\times 2\sqrt{2\pi}[1-Q(3)]$$
$$= 10\times 2\sqrt{2\pi}[1-0.1350\times 10^{-2}]$$
$$= 10\times 2\sqrt{2\pi}\times 0.99865$$
$$= 50.06$$

9.2 Equvalent Noise Bandwidth

We know that the system output power

$$R_{YY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)|H(\omega)|^2e^{j\omega\tau}\,d\omega=\int_{-\infty}^{\infty}S_{XX}(f)|H(f)|^2e^{j2\pi f\tau}\,df \quad (9.3)$$

The power of the output process

$$P_{YY}=R_{YY}(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)|H(\omega)|^2\,d\omega \quad (9.4)$$

The equation (9.4) is often used in practical applications. But we need simplified calculation method to compute the noise power at the output of a filter.

Let $H(\omega)$ is the lowpass system transfer function and the spectrum of the input process equals to $\frac{N_0}{2}$ for all ω, with N_0 a posite, real constant (such spectrum is called a white noise spectrum).

By using equation(9.4),

$$P_{YY}=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\frac{N_0}{2}|H(\omega)|^2d\omega$$

Let define ideal LPF $H_I(\omega)=\begin{cases}H(0); & |\omega|\le\omega_N\\ 0; & |\omega|>\omega_N\end{cases}$

where W_N is a positive constant chosen such that the noise power at the output of the ideal filter is equal to the noise power at the output of the original (practical) filter.

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_0}{2}|H(\omega)|^2\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N_0}{2}|H(0)|^2\,d\omega \quad (9.5)$$

If $|H(\omega)|^2$ to be an even function of ω, then

$$W_N = \frac{\int_0^\infty |H(\omega)|^2 \, d\omega}{|H(\omega)|^2} \quad (9.6)$$

where W_N is called the equivalent noise bandwidth of the filter with the transfer function $H(\omega)$.

In the above Fig., the solid curve represents the practical characteristic and the dashed line the ideal filter rectangular one.

The equivalent noise bandwidth is such that in this picture the dashed area is equalsthe shared area.

From equation (9.4) and (9.5), the output power of the filter can be written as

$$P_{YY} = \frac{N_0}{2\pi} |H(0)|^2 W_N \quad (9.7)$$

Thus, it can be shown for the special case of white noise input that the integral of equation (9.4) is reduced to a product and the filter can be characterized by means of a single number W_N as far as the noise filtering behavior is concerned.

9.3 Thermal Noise

- Thermal noise is produced by thye random motion of electrons is a medium.
- The intensity of this motion increases with increasing temparature and is zero only at a temparature of absolute zero.
- If the voltage accross a resistor is examined using a sensitive oscilloscope, a random pattern will be displayed on the screen. The PSD of this r.p is

$$G(f) = \frac{A(f)}{e^{B|f|} - 1}$$

  where A and B are constant depend on temperature and other physical constants.
- For the frequencies below the knee of the curve, $G(f)$ is almost constant. If we operate in this frequency range, wewe can consider thermal noise to be white noise.
- In fact, thermalnoise appears to be approximately white upto extremely high frequencies, i.e., $10^{13}\ Hz$. For frequencies with in this range, the mean square value of the voltage across the resistor ($R[0]$) has benn shown to equal.

$$\overline{v^2} = R(0) = 4KTRB$$

where $K-$ Boltzmens constant $= 1.38 \times 10^{-23}\ J/K$
$T-$ Temperature in ^{o}K
$R-$ resistance value
$B-$ observation bandwidth
$\therefore$ The hight of PSD over this constant region is $2KTR$.

Q: What is the power generated by a resistor, that is if a resistor is connected to a additional circuit? How much noise power is generated in additional circuit?
A: From basic circuit theory, the circuit depends on the impedance of the external circuit, and the power transferred is a maximum when the load impedance matches the generater impedence. This yields the maximum available power, which (using a voltage divider relationship) is

Maximum available power $\boxed{N = \frac{V^2}{4R} = KTB}$

with corresponding PSD of $\boxed{G_N(f) = \frac{KT}{2}}$

$\therefore$ $G_N(f)$ is the PSD of the available noise power from a resistor.

- If we have a system with number of noise generating devices with in it, we often refers to the system noise temperature, T_e in Kelvins. This is temperature of a single noise source that would produce the same total noise power at the output.
- If the input to the system contains noise, the system then adds its own noise to produce larger output noise.
- The system noise figure is the ratio of noise power at the output to that at the input. It is usually expressed in decibels.
  EX: If the noise figure is $3\ dB$ indicates that the system is adding an amount of noise is equal to that which appears at the input. So, the output noise power is twice that of the input.
- For thermal noise,
  - most communication system will operate below $100\ MHz$
  - Noise of the communication system $< 100\ MHz$ and it consists finite power. For example,
  - Voice freq: $300 - 3400\ Hz \approx 0$ to $4\ KHz$
    sampling freq : $8\ KHz$ each 8-bits
    Speech rate per sample: $64\ Kbps$
  - AM freq: $550 - 1600\ KHz \Longrightarrow$ broad casting
  - FM radio: $88 - 108\ MHz$

9.4 Narrow Band Noise

- Most communication system deals with band pass filters. Therefore while noise appearing at the input to the system will be shaped into band limited noise by the filtering operation. If the bandwidth of the noise is relatively small compared to the center frequency. We refer to this as narrow band noise.
- We have no problem deriving the PSD and ACF of this noise, and these quantities are sufficient to analyze the effect of linear system. However, by dealing with multipliers and the frequency analysis approach is not sufficient, since non-linear operations are present. In such cases, it proves useful to have trigonometric expressions for the noise signals. The form of this expressions is

$$n(t) = x(t)\cos 2\pi f_0 t - y(t)\sin s\pi f_0 t \tag{9.8}$$

where n_0 is noise waveform and f_0 is center frequency the band occupied by noise.

The sine and cosine vary by 90 degrees, $x(t)$ and $y(t)$ are known as the quadrature components of the noise.

- From equation (9.8), we derive from exponential notation

$$n(t) = Re\{r(t)e^{j2\pi f_0 t}\}$$

where $r(t)$ is a complex function with a low frequency band limited fourier transform. Re is the real part of the expression in the brackets that folls it, and the exponential function has the effect of shifting the frequencies of $r(t)$ by f_0. By Euler's identity,

$$r(t) = x(t) + jy(t) \tag{9.9}$$

$$\therefore\ n(t) = Re\{[x(t) + jy(t)](\cos 2\pi f_0 t + j\sin 2\pi f_0 t)\}$$

$$n(t) = x(t)\cos(2\pi f_0 t) - y(t)\sin(2\pi f_0 t) \tag{9.10}$$

The equation (9.8) and (9.10) are equal. But equation (9.8) is not simple way to do by using "Hilbert transforms".

9.4.1 Hilbert Transforms

- The Hilbert tranform of a function of time is obtained by shifting all frequency components by -90^o.

- The Hilbert transform operation can be represented by a linear system, with $H(f)$ as shown in Fig.

- The phase function of a real system must be odd. The system function is given by

$$H(f) = -j\,\mathrm{sgn}(f)$$

- The impulse response of this system is inverse tranform of $H(f)$. This is given by

$$h(t) = \frac{1}{\pi t}$$

- The Hilbert tranform of $S(t)$ is given byconvolution of $S(t)$ with $h(t)$. Let us denote the transform by $\hat{S}$, then

$$\therefore \hat{S} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{S(\tau)}{t-\tau}\, d\tau$$

If we take the Hilbert transform of Hilbert transform, the effect in frequency domain is to multiply the transform of the signal by $|H^2(f)|$. But $H^2(f) = -1$, so we retain to the original signal which is change of sign. This indicates that the inverse Hilbert transform equation is the same as the transform relation-ship, except with a minus sign.

$$\therefore S(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{S}(\tau)}{t-\tau}\, d\tau$$

Q: Find the Hilbert transform of the following time signals

(a) $S(t) = \cos(2\pi f_0 t + \vartheta)$

(b) $S(t) = \frac{\sin 2\pi t}{t} \cos 200\pi t$

(c) $S(t) = \frac{\sin 2\pi t}{t} \sin 200\pi t$